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UNIVERSITY OF ALBERTA

A PROPOSED SOURCE FOR THE KERR BLACK HOLE GEOMETRY AND THE GEODESIC PROPERTY OF COSMIC STRINGS

BY

DESMOND JOHN MC MANUS

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

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DOCTOR OF PHILOSOPHY

IN

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Date: September 20, 1991

Boyle: *An', as it blowed an' blowed, I ofen looked up at the sky an' assed meself the question —what is the stars, what is the stars?*

Joxer: *Ah, that's the question, that's the question —what is the stars?*

—SEAN O'CASEY, *Juno and The Paycock* (1925)

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommended to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "A proposed source for the Kerr black hole geometry and the geodesic property of cosmic strings" submitted by Desmond John Mc Manus in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Theoretical Physics.

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ABSTRACT

A source for the Kerr black hole geometry is presented. The source consists of a toroidal shell that encloses the ring singularity, a disc that spans the equatorial plane and an annulus of matter interior to the torus. The torus is also filled with some matter that gives negligible contributions to the overall mass and angular momentum of the source. The disc and annulus contribute negative masses and angular momenta. The toroidal shell gives positive values. The source as a whole gives the correct values for the mass and angular momentum of a Kerr black hole.

The work on the Kerr black hole is extended to the Kerr-Newman black hole. The source for the Kerr-Newman black hole geometry is very similar to that for the Kerr black hole. Additional care must be taken to account for the effect of the electromagnetic field. The source gives the correct values for the mass, angular momentum, and electric charge of a Kerr-Newman black hole.

Spacetime splittings are also examined. The ADM 3+1 spacetime split is extended to a generalised, $m+n$, splitting of spacetime. The global spacetime has dimension $(m+n)$ and the foliating spaces have dimension m. There are n independent normals to each of these foliating spaces, thus giving n distinct extrinsic curvatures. The generalised Gauss-Weingarten and the generalised Gauss-Codazzi equations associated with this splitting are derived. Both these sets of generalised equations are shown to reduce to the familar ADM equations when a 3+1 split is considered. In the case of an orthogonal splitting, the generalised equations are found to exhibit a particularly elegant form.

Finally, the geodesic properties of cosmic strings are investigated. It is shown that idealised cosmic strings deform the background geometry in such a

way that they become geodesics of spacetime. For deficit angles less than π, the background curvature becomes unbounded as the string is approached. However, the divergence is quite weak. These general findings are backed up by two different families of cosmic string solutions.

PREFACE

The Faculty of Graduate Studies and Research currently accepts two formats for thesis preparation: traditional format, and paper format. This thesis is presented in paper format.

Chapter one consists of review material on black holes and cosmic strings, along with an outline of the thesis. The material in chapter two is based on a paper titled "A Toroidal source for the Kerr black hole geometry."[1] Chapter three extends the work of chapter two to the Kerr-Newman black hole. A general formalism for dealing with an arbitrary splitting of spacetime is developed in chapter four. Chapter five is based on a paper titled "Cosmic strings are straight"[2]; this paper was written in collaboration with Unruh, Hayward, and Israel. The final chapter sums up the main results of the thesis.

Four appendices are also included. Appendices A, B, and C, contain information necessary for performing some of the calculations in chapters two and three. Appendix D contains the full form of a result mentioned in chapter four.

---

[1] Mc Manus, D.J. (1991), *Class. Quant. Grav.*, **8**, 863.

[2] Unruh, W.G., Hayward, G., Israel, W., and Mc Manus, D. (1989), *Phys. Rev. Lett.*, **62**, 2897.

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I thank my parents, Des and Fran Mc Manus, for their continual support and encouragement. They were always there when I most needed them. I especially thank them for fostering a wonderful learning environment, and for always showing a genuine interest in my schooling. I honestly do not think I would have reached this stage if they had not been behind me all the way. Their love and

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Most important of all, I thank my wife, Catherine Smith, for her love and friendship. She has always supported me in my endeavor. Her encouragement was indispensible in helping me finish, and I really appreciate the sacrifices that she had to make on my behalf. Foremost, I cherish her friendship.

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CHAPTER ONE

INTRODUCTION

1.1 Introduction

This thesis examines several problems in relativity: the source of the Kerr and the Kerr-Newman black hole geometries, generalised splitting of spacetime, and the geodesic property of cosmic strings.

The customary function of the introduction is to link the main topics of the thesis in some sort of coherent fashion. This is not an easy task for this thesis as can be seen from the topics mentioned in the opening paragraph. The main connection that threads the chapters together is technical. Loosely speaking all the chapters involve the embedding of hypersurfaces in one way or another. Briefly some examples are as follows: a main component of the source of the rotating black hole geometry is a disc of matter; the connection to generalised splitting of spacetime is self evident; and finally, the world-sheet of an idealised cosmic string can be modeled as a two dimensional surface whose points are conical singularities of the spacetime. Techniques involving hypersurfaces are used extensively throughout the thesis.

The remainder of this chapter consists of review material. The main aim is to provide the reader with a general background for the topics that are covered in this thesis. For the most part each section will be general in nature since each chapter of the thesis is supplied with its own detailed introduction.

Section two consists of background information on black holes with specific

emphasis on the properties of the Kerr black hole. Section three deals with the formation of cosmic strings and their physical properties. Some time is spent examining straight cosmic strings because they provide a good basis for examining other string configurations. The end of this chapter outlines the main results of the thesis.

1.2 Black Holes

The idea of black holes has been around for slightly over two hundred years. Rev. Michell[1] (1784) was the first to discuss the idea of stars so massive that not even light could escape. Michell's idea was formulated in the framework of Newtonian theory and the corpuscular theory of light. Laplace[2] (1796) also worked on this idea a short time later. The advent of Young's wave theory of light quenched further speculations on the deflection of light by gravitational bodies and the concept of black holes was to lie dormant for the next century.

Even with the advent of Einstein's general theory of relativity[3] in 1915, and the subsequent discovery of the Schwarzschild solution[4] shortly thereafter, the concept of black holes was not openly welcomed by the physics community. The main reasons for the tardy acceptance of black holes were probably the lack of direct physical evidence for their existence and the then outrageous idea of a very large mass confined to a small region of space thus producing "unnaturally" high densities.

During the next forty years or so, there were sporadic attempts to probe the nature of the spacetime region beyond the Schwarzschild horizon. Black holes began to secure a footing during the sixties especially with the observation of quasars and compact X-ray sources. Indeed the phrase "black hole" was not

coined until 1968 by Wheeler[5]. The main advances in black hole physics since the sixties are the singularity theorems of Penrose and Hawking[6,7,8], the "no hair" theorems, and black hole thermodynamics.

The presence of singularities in gravitational collapse was initially thought to be a manifestation of the high degree of spherical symmetry of the models under consideration. The belief was that if perturbations were applied then the singularity could be avoided[9,10]. However, the singularity theorems proved that the presence of a trapped surface always implied the presence of some sort of spacetime singularity thus establishing that singularities are generic features of gravitational collapse. A word of caution should be added here: the appearance of singularities should be perceived as indicating a breakdown in the theory of general relativity. Quantum physics comes into play near the singularity and a full quantum theory of gravity is needed to comprehend the nature of the singularity.

Several black hole solutions are known: the Schwarzschild solution which represents a static spherically symmetric massive black hole, the Reissner-Nordström[11,12] solution which is the charged analogue of the Schwarzschild black hole, the Kerr[13] solution which describes a massive rotating black hole, and finally the Kerr-Newman[14] solution which is the charged analogue of the Kerr black hole. A series of uniqueness theorems was proved by Israel[15,16], Carter[17], Robinson[18], and Mazur[19] which culminated in the result that the Kerr-Newman black holes are the only allowable stationary electrovac black holes. These theorems are collectively referred to as the "no hair theorems". Almost all the multipoles are radiated away during gravitational collapse. The fields radiate gravitationally and electromagnetically (if a charge or magnetic field is present) while settling down, so that eventually the only information that survives is the mass, angular momentum, and charge of the black hole.

We will concern ourselves with the sources of the Kerr and the Kerr-Newman black hole geometries in chapters two and three. Therefore, at this point it is worthwhile commenting on the internal structure of black holes. The Schwarzschild metric accurately describes the spacetime geometry exterior and interior to the Schwarzschild horizon; however, the same cannot be said about the Kerr metric. The Kerr metric possesses two horizons, an inner and an outer horizon. The inner horizon is also a Cauchy horizon. Light rays falling into the black hole accumulate at the inner horizon. There is an infinite accumulation of light rays at the inner horizon which results in a singular surface of infinite blue shift at the inner horizon. The Cauchy horizon is unstable; the spacetime curvature becomes infinite at the Cauchy horizon developing a true singularity and thus making the region interior to the inner horizon inaccessible[20]. Quantum considerations impose an upper bound on the curvature thus allowing the region interior to the Cauchy horizon to be determined by quantum effects.

An interior Schwarzschild solution is known: A static spherically symmetric perfect fluid can act as a source for the Schwarzschild geometry. This is encouraging since we associate black holes with the end product of gravitational collapse of stellar objects. Unfortunately no such rotational analogue has been found for the interior Kerr metric. We expand on this in greater detail in the introduction to chapter two where we propose a source for the Kerr black hole geometry.

The reader should be aware that our analysis of a possible source for the Kerr and Kerr-Newman black hole geometries is highly idealistic, as quantum considerations are overlooked. We are also aware that since our sources may lie within the inner horizon, they may not be physically relevant. We adopt the standpoint that our classical models may throw some light on the problem of the source of the geometry of rotating black holes and offer some new leads on how to

tackle the problem in its entirety.

The Kerr black hole

A two parameter family of rotating black hole solutions was found by Kerr in 1963. The Kerr metric in Boyer-Lindquist coordinates[21] $(\bar{t}, r, \theta, \bar{\phi})$ is

$$\begin{aligned} ds^2 &= \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2)\sin^2\theta\, d\bar{\phi}^2 - d\bar{t}^2 \\ &\quad + \frac{2mr}{\Sigma}(a\sin^2\theta\, d\bar{\phi} + d\bar{t})^2 \end{aligned} \tag{1.1}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2mr$, and where m and a are constants; m is the mass and $-ma$ is the angular momentum of the black hole as measured from infinity. The metric is singular at $\Delta = 0$ which is a coordinate singularity and at $\Sigma = 0$ which is a curvature singularity. Solving $\Delta = 0$ for r we find two solutions, $r_\pm = m \pm \sqrt{m^2 - a^2}$, where the plus and minus subscripts represent the outer and inner horizons, respectively. The metric can be analytically continued across each horizon, but it is only possible to find a coordinate patch that covers one or the other of the horizons. It is impossible to find a coordinate patch that includes both horizons. The curvature singularity, $\Sigma = 0$, corresponds to $r = 0$, $\theta = \pi/2$. This singularity is actually a ring singularity which can most readily be seen by transforming to Kerr-Schild coordinates (x, y, z, t) where

$$x + \mathrm{i}\,y = (r + \mathrm{i}\,a)\sin\theta\exp\{\mathrm{i}\int(d\bar{\phi} + \frac{a}{\Delta}dr)\} \tag{1.2}$$

$$z = r\cos\theta \tag{1.3}$$

$$t = -r + \int(-d\bar{t} + \frac{r^2 + a^2}{\Delta}dr)\ . \tag{1.4}$$

In these coordinates r is determined implicitly, up to sign by

$$r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2z^2 = 0\ . \tag{1.5}$$

For $r = 0$, $\theta = \pi/2$ we find that now $x^2 + y^2 = a^2$, $z = 0$ which is the boundary of a disc.

A further disparity between the Kerr and the Schwarzschild metrics is the behaviour of geodesics. All timelike and null geodesics of the Schwarzschild metric eventually end up at the singularity once they cross the Schwarzschild horizon; whereas, only timelike and null geodesics which are confined to the equator can reach the Kerr ring singularity.

1.3 Cosmic Strings

Kibble[22] pioneered the cosmological role of cosmic strings. In his 1976 paper, he studied the possible domain structures which can arise in the universe in a spontaneously broken gauge theory. Possible structures include domain walls, cosmic strings, and monopoles. Domain walls and monopoles that occur in gauge theories with a symmetry breaking scale of about 10^{16} GeV lead to cosmological disasters; they over-close the universe. The gravitational field of domain walls would be unacceptably large and the predicted over-abundance of monopoles has not been observed. Therefore, theories which allow the formation of domain walls and monopoles should probably be ruled out. On the other hand cosmic strings do not apparently cause any cosmological disasters and may have some interesting cosmological consequences[23,24].

Cosmic strings are regions of trapped false vacuum that can arise during phase transitions in the early universe. As the temperature of the universe cools below a critical temperature, the Higgs field develops a non-zero expectation value $<\phi>$. The values of $<\phi>$ are uncorrelated on distances greater than the horizon scale, that is, distances travelled by light during the lifetime of the universe.

Topological defects occur on the interfaces between regions with different values of $<\phi>$. The defects arise when a gauge symmetry is broken. Cosmic strings form when the manifold of equivalent vacua, $\mathcal{M}$, is not simply connected. In other words $\mathcal{M}$ must contain unshrinkable curves. A prime example is when a local U(1) symmetry is broken. Strings cannot have ends; they either form closed loops or extend infinitely.

The physical attributes of strings are related to the energy scale at which the symmetry is broken. We can estimate the thickness, δ, and the mass per unit length, μ, of strings from the energy scale, E, of the symmetry breaking. On dimensional grounds we find

$$\mu = \frac{E^2}{\hbar c^3} \tag{1.6}$$

$$\delta = \frac{\hbar c}{E} \ . \tag{1.7}$$

For a grand unification scale, $E \sim 10^{16}$ GeV, we get $\delta \sim 10^{-30}$ cm and $\mu \sim 10^{22}$ g/cm (with $\hbar = 1.03 \times 10^{-27}\,\mathrm{cm^2\,g\,s^{-1}}$, $c = 3 \times 10^{10}\,\mathrm{cm\,s^{-1}}$, and 1 eV $= 1.6 \times 10^{-12}\,\mathrm{cm^2\,g\,s^{-2}}$). Strings of cosmological interest have lengths much greater than than their width, and we can approximate them by infinitely thin curves.

a) Straight Cosmic Strings

To obtain some insight into the fascinating properties of cosmic strings, we shall examine the geometry of the simplest string solutions, namely, the straight static string. The internal structure of the string is unimportant, and we can average out the physical quantities over the string cross-section. Thus we are able to write the energy-momentum tensor of an infinitely long static string lying along the z-axis

as

$$T_{\mu}^{\ \nu} = -\mu\,\delta(x)\,\delta(y)\,(\delta_{\mu}^{t}\,\delta_{t}^{\nu} + \delta_{\mu}^{z}\,\delta_{z}^{\nu}) \tag{1.8}$$

where μ is the linear energy density. We observe that the tension along the string is equal to the mass per unit length of the string. This equation of state is characteristic of a false vacuum, $P = -\rho$. The string solution is invariant under Lorentz boosts along the z-direction and so only transverse motions of the string are detectable. We assume that the metric has the form

$$ds^2 = A^2(\rho)(-dt^2 + dz^2) + d\rho^2 + B^2(\rho)\,d\phi^2 \quad . \tag{1.9}$$

Outside the string core the space is empty and therefore the Kasner metric[25] describes the geometry since the metric only depends on one spatial variable. The Kasner metric is

$$ds^2 = -\rho^{2p_1}\,dt^2 + d\rho^2 + \rho^{2p_2}\,d\phi^2 + \rho^{2p_3}\,dz^2 \tag{1.10}$$

where p_1, p_2, p_3 are subject to the constraints

$$p_1 + p_2 + p_3 = (p_1)^2 + (p_2)^2 + (p_3)^2 = 1 \quad . \tag{1.11}$$

Comparing the two results we see that the only two possible solutions are

$$i)\ \ p_1 = p_3 = 0, \ \ p_2 = 1 \tag{1.12}$$

$$ii)\ \ p_1 = p_3 = \tfrac{2}{3}, \ \ p_2 = -\frac{1}{3} \quad . \tag{1.13}$$

The latter solution is ruled out on physical grounds since the length of a circle $\rho = const.$ is proportional to $\rho^{-1/3}$ and this length tends to zero as ρ tends to infinity. Thus, the space outside the string is flat and described by the metric

$$ds^2 = -dt^2 + d\rho^2 + \rho^2\,d\phi^2 + dz^2 \quad . \tag{1.14}$$

The spacetime is not however globally equivalent to Minkowski spacetime as can be seen from the exact vacuum string solution of Gott[26]. (Hiscock[27] and Linet[28] independently found the same result.) Gott obtained exact interior and exterior solutions to Einstein's field equations for vacuum strings. The interior metric is

$$ds^2 = -dt^2 + dr^2 + r_0^2 \sin^2(\frac{r}{r_0})\, d\phi^2 + dz^2 \tag{1.15}$$

with $-\infty < t, z < \infty$, $0 \leq \phi < 2\pi$, $0 \leq r \leq r_0\,\theta_M$ where both r_0 and θ_M are constants. The only non zero components of the energy momentum tensor are

$$T_t^t = T_z^z = \frac{-1}{8\pi r_0^2} \tag{1.16}$$

which corresponds to the desired equation of state, $P_z = -\rho$. The exterior solution is

$$ds^2 = -dt^2 + d\bar{r}^2 + (1-4\mu)^2\,\bar{r}^2\, d\phi^2 + dz^2 \tag{1.17}$$

where t, z, ϕ have the same range as before and $\frac{r_0}{1-4\mu}\sin\theta_M \leq \bar{r} < \infty$.

The metric is continuous across the boundary between the interior and the exterior solutions. The interior boundary is located at $r = r_0\,\theta_M$ and the exterior boundary is situated at $\bar{r} = \frac{r_0}{1-4\mu}\sin\theta_M$. The extrinsic curvatures are also continuous across the boundary which guarantees that the boundary is not a surface layer.

The coordinate transform $\phi' = (1-4\mu)\phi$ reduces the line element (1.17) to its Minkowskian form. The solution represents a conical singularity since $0 \leq \phi' \leq 2\pi(1-4\mu)$. The resulting space has an azimuthal deficit angle, $\Delta\phi = 8\pi G\mu$. The deficit angle can be visualised as cutting a wedge out of a disc and identifying opposite edges to form a cone. In the limit $\mu \ll 1$ the metric reduces to the weak field solution found by Vilenkin[29].

b) Physical effects due to cosmic strings

Cosmic strings produce many gravitational effects that may be observable: gravitational radiation, double images, and a steplike discontinuity in the cosmic microwave background(CMB).

Closed loops oscillate at exceedingly high rates due to the extreme tensions involved. Gravitational radiation is the main source of energy loss for non-intersecting loops. Oscillating loops produce a residual stochastic gravitational radiation background with a distinctive logarithmic spectrum[30,31,32]. However, at present this background is below observational limits.

Even though spacetime is flat outside straight cosmic strings, gravitational lensing effects are still possible due to the conical nature of the space. The most notable effect is the double image of quasars and galaxies[26,29,33]. A very distinctive signature for cosmic strings would be the observation of a line of double images.

Relativistic strings produce a Doppler shift in the frequency of light rays that pass on different sides of the string. This Doppler shift is manifest as a temperature discontinuity in the CMB, often referred to as the Kaiser-Stebbins effect[34]. The observed lack of anisotropy of the CMB ($\frac{\delta T}{T} < 10^{-4}$) on angular scales greater than 10^0 imply $G\mu < 10^{-5}$ if any strings exist.

One of the most salient features of cosmic strings is their proposed role in the formation of large scale structure in the universe[35,36]. Cosmic strings might act as seeds for galaxy formation. Since the initial proposal of the role of strings in galaxy formation, the actual proposed mechanism has changed somewhat. The current belief[37] is that wakes are more important than loops. Planar wakes form

behind relativistic moving strings. Most galaxies and clusters must form from fragmentation of these wakes. Loops still however play a reduced role — those not bound to strings may act as galaxy seeds.

1.4 Outline of thesis

Several different problems are tackled in this thesis.

The first problem we assault is the problem of finding a source for the Kerr black hole geometry. Many attempts have been made since the discovery of the Kerr metric to match the exterior Kerr metric to an interior metric that would act as a source for the exterior geometry. Initial investigations concentrated on finding perfect fluid sources and later work proposed that the source consisted of a ring or a disc of matter spanning the ring. We give a historical account of this problem, outlining the three main approaches to the problem: (i) trying to find a perfect fluid interior; (ii) searching for a spinning shell or spheroidal solution; and (iii) examing possible disc-like or ring sources. We argue against the first two approaches and pursue the third approach.

Our approach[38,39] is based on Israel's[40] line source approach. Israel suggested that the Kerr ring could be modeled as a line source. Close up the ring can be taken to be a straight line, at least locally. The line is then enclosed by a cylinder such that the geometry exterior to the cylinder is given by the Kerr metric. A new metric is chosen interior to the cylinder which has a milder behaviour as the singularity is approached. The interior region also turns out to be a vacuum region in the line source approximation. The source is then a layer of matter distributed over the cylinder's surface. The cylinder is actually shrunk onto the ring to form the source. Our work was motivated by the claims of

López[41] and Taub[42] which stated that Israel's method did not give the correct mass and angular momentum for the Kerr metric. We extend Israel's earlier work by enclosing the ring in a toroidal tube and picking an interior metric. However, we do not make any line source approximations but rather examine the source in full detail. Our proposal for the source of the Kerr black hole geometry consists of a toroidal shell that encloses the ring singularity, a disc that spans the equatorial plane, and an annulus of matter interior to the toroidal shell. We find that our source gives the correct mass and angular momentum.

In chapter three, we extend our findings of chapter two to the source of the Kerr-Newman black hole geometry. The source, as might be expected, is very much the same as the source of the Kerr geometry. Now however care must be taken to account for the electromagnetic field. The electromagnetic field contributes to the mass and angular momentum of the source via the the electromagnetic energy density and the Poynting current. We find that the electromagnetic field is non zero in the interior of the source, and the major contributions from the interior to the mass and angular momentum come from the electromagnetic field.

Chapter four is concerned with spacetime splitting. Usually spacetime splitting is thought of as meaning that spacetime is foliated into a family of non-intersecting timelike or spacelike hypersurfaces. This approach was pioneered by Arnowitt, Deser, and Misner (ADM)[43] and is commonly referred to as the ADM 3+1 split. Their idea is that the geometry of the four-manifold is described in terms of the intrinsic metric and the extrinsic curvature of a three dimensional hypersurface, along with the lapse function and the shift vector. The lapse and the shift relate the intrinsic coordinates on one hypersurface to the intrinsic coordinates on a nearby hypersurface. We generalise the ADM idea of spacetime splitting by considering a foliation where the dimension of the foliating spaces is

taken to be arbitrary. Each foliating space will in general have n independent normal vectors with which we associate extrinsic curvature tensors thus giving us n generalised extrinsic curvature tensors. We also require a set of n shift vectors and a lapse matrix to relate the intrinsic coordinates on the different foliating spaces. We essentially study the properties of submanifolds. These properties have in the past been studied in a pure mathematical context[44,45,46]. Our goal is to examine the properties of the foliating spaces in a language that is readily accessible to physicists, especially those already familiar with the ADM 3+1 split.

Formally we consider a $m+n$ splitting of spacetime where m is the dimension of the foliating spaces. We generalise the ADM quantities along with introducing some new ones. The full derivation of the generalised Gauss-Weingarten and the Gauss-Codazzi equations is given, and the ADM 3+1 split is related to a special case of our formalism.

In chapter five, we analyse the properties of idealised cosmic strings. We investigate the allowable configurations of idealised cosmic strings. The results of Clarke, Ellis, and Vickers (CEV)[47] suggested that strings are inflexible and that strings cannot bend on length scales small compared to the cosmological horizon. This rules out the ability of strings to form closed loops which would in turn partially alter the role of cosmic strings in galaxy formation. CEV's findings also conflict with the work of Frolov, Israel, and Unruh (FIU)[48]. A family of momentarily stationary circular loops was found by FIU. The main feature of the FIU solutions that we wish to emphasise is that FIU's family of solutions consist of loops of arbitrary radii and deficit angles which is counter to CEV's claim. We investigate the apparent contradiction between the results of CEV and FIU.

We model the world sheet of an idealised cosmic string as a two-surface

whose points are conical singularities of spacetime[49]. Firstly, we are able to show that intrinsic geodesics on the world sheet are automatically geodesics of the full spacetime. We interpret CEV's claim of rigidity as meaning that cosmic strings are geodesics. Secondly, we investigate the behaviour of the Riemann tensor in the vicinity of the world sheet and find that for deficit angles less than π the magnitude of the Riemann tensor diverges as the string is approached. However, this divergence is quite weak. We back our general findings up with specific examples from the two classes of the FLRW solutions.

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CHAPTER TWO

A TOROIDAL SOURCE FOR THE KERR BLACK HOLE GEOMETRY

2.1 Introduction

Since the Kerr metric[1] was first discovered, many people have attempted to find an internal solution that would act as a source. Kerr claimed that the metric represented the gravitational field exterior to a rotating body. It was not long before it was shown that indeed the parameters "m" and "a" represented the mass and the angular momentum per unit mass of the source, respectively; (Boyer and Price[2], Cohen[3], Léauté[4], Moss and Davis[5]).

The search for an interior solution for the Kerr metric can be categorised into three main approaches: (*i*) trying to find a perfect fluid interior; (*ii*) looking for a spinning shell or spheroidal solution; and (*iii*) examining possible disc-like or ring sources. We support the work of the researchers in the third category but before discussing our work it is useful to present a review of the previous efforts to find an interior Kerr solution.

Initially, a perfect fluid interior solution was sought since it seemed the most natural choice. Boyer[6] examined possible perfect fluid sources and he derived the equation for the only possible fluid boundaries compatible with the requirement that the fluid pressure was zero on the boundary. Hernandez[7] followed with an argument that indicated that a perfect fluid interior may not exist. However, Hernandez's interpretation was considered to be incorrect, his work did not disprove the existence of a perfect fluid interior, as he had concluded, but rather proved

that his initial guess for an interior metric was wrong, see Krasiński[8]. Roos[10] proved that, at least locally, we cannot rule out the possibility of a perfect fluid source in an arbitrary small neighbourhood of a possible boundary. Abramovicz *et al.*[11] employed the relativistic virial theorem and the Raychadhuri equation to extend the work of Boyer[6]; however, their conditions were not sufficient to rule out the existence of a rigidly rotating fluid source. De Felice *et al.*[12] investigated the Boyer condition. They found closed spheroidal perfect fluid configurations with an internal cavity and toroidal configurations, but they judged these configurations to be non-physical. Abramowicz *et al.*[13] excluded the possibility of a pressureless disc on the assumption that the disc must be a closed surface, but as Krasiński[8] indicated, this need not be true. Whereas none of these papers excluded the possibility of a perfect fluid interior, there is no apparent reason to believe that there should be one, except on the grounds of aesthetics and we suspect that the answer probably lies elsewhere. Further suggestions have proposed a non-perfect fluid interior solution, that matches with the Kerr solution on a closed surface that lies outside the event horizon[21], and that an interior solution may be discovered by examining ellipsoidal spacetimes[8] . Also an anisotropic fluid interior was found by Gürses and Gürsey[9], that matched with the exterior Kerr metric on an oblate spheroid.

Another early approach was to look for spinning spheroids or shells. Cohen[3] examined the case when the angular momentum per unit mass of the source is very small and he found that a thin slowly rotating mass shell could act as a source of the Kerr metric. This idea was furthered by De la Cruz and Israel[14]. They investigated the properties of a slowly spinning, nearly spherical, thin shell that has the Kerr metric as the exterior field. Florides[15,16] used the Florides-Synge[17,18] method of successive approximations to match the Kerr metric approximately

to a steadily rotating sphere and later, a rotating spheroid of small eccentricity. Mc Crea[19] employed a technique developed by Synge[20] to examine the gravitational field of a uniformly rotating sphere and showed that it agreed with the Kerr metric to a third order approximation. The solutions offered [3,14,15,16,19] are approximations which give us some insight, but do not encompass the full range of possible Kerr metrics.

The idea of a spinning ring has been offered as a source of the Kerr metric by many researchers. Based on several characteristics of the Kerr metric, Janis and Newman[22] suggested that the source is a ring of matter rotating about its axis of symmetry. In a footnote, they add that they may also need a surface distribution of matter with the ring as a boundary. Keres[23] reduced the relativistic problem to a problem in Newtonian potential theory. In his analysis the Kerr gravitational field is generated by a disc-like surface layer of negative mass bounded by a ring of positive mass with an infinite linear density. By considering inertial effects Hernandez[24] inferred that rigidly rotating rings of mass could not serve a source. Israel[25] considered the Kerr-Newman metric as arising from a layer of mass and charge distributed over the equatorial disc spanning the ring singularity. The lack of smoothness of the metric at $(r = 0)$ is assumed to be caused by a layer of mass spread over the disc—there is a discontinuity in the normal derivative of the metric across the flat disc. The physical properties of the disc can easily be obtained by applying the theory of surface layers, see Israel[26]. Then, by using a "subtraction argument" the nature of the ring singularity can be inferred. The deduction was made that the ring must have positive infinite mass and infinite positive angular momentum so as to yield the correct mass and angular momentum. Hamity[27] found a similar result, that the Kerr solution can be attributed to a thin rigidly rotating disc with a regular interior and a singular rim; the stress-energy tensor

on the disc was attributed to some sort of fictitious material. Israel[28] examined the source structure of the ring singularity using the technique of line sources. He found it to consist of dust-like material whose mass distribution had a "demipole" structure circulating around the ring with the speed of light. Taub[29] used a distribution formalism for hypersurfaces to re-examine Israel's[28] line source. He concluded that, "the assumptions made by Israel in his discussion of the Kerr metric do not lead to correct values for both the mass and angular momentum of the Kerr metric." It should be noted that Israel's assumptions do actually give the correct values. We shall comment on this fact in section six of this chapter. A year later, López[30] used the mathematical theory of distributions to investigate the source of the Kerr geometry. He dismissed the possibility of a pure line source on the strength of Taub's argument. A stress-energy tensor was obtained that had its support throughout the disc and along the ring. He then showed that this stress-energy tensor gave the right values for the mass and angular momentum. The total mass and angular momentum were both made up from contributions from the disc and ring. The form of these contributions is exactly predicted by a slight modification of Israel's line source solution. Grøn[31] later noted, "the seemingly different sources of Israel[25] and Hamity[27] are, in fact, identical." Hamity and Lamberti[32] ruled out a family of rotating discs, on the grounds that none of the discs satisfy the dominant energy condition and in some instances the weak energy condition is also violated. They seemed to think that their sources are not physically significant.

Our approach examines possible disc-like sources, and is in strong agreement with some earlier findings [25,28,27,30,31,32]. The purpose of this chapter is to find a source of the Kerr geometry that is well defined for any values of the mass and angular momentum. As quite a few papers have shown, it seems unlikely that

the source of the Kerr geometry is a perfect fluid. The problem we are then faced with is: what stress-energy tensor should we pick? We circumvent this problem by the old "Synge trick"; instead of guessing the form of the stress-energy tensor and then trying to solve the Einstein field equations, we take the alternative approach of defining the stress-energy tensor in terms of the Einstein tensor, that is, we define the geometry by specifying a metric and the stress-energy tensor is just given by $T_{ij} = G_{ij}/8\pi$. Our aim is to choose a metric that is reasonably well-behaved as we approach the ring, at least better behaved than the Kerr metric, and to see what mass and angular momentum distribution it implies. We consider this method of letting the geometry dictate the stress-energy tensor as equally valid as the reciprocal approach for the problem at hand.

Our source consists of a toroidal tube surrounding the ring singularity, see figure 2.2. The tube is filled with some appropriate matter. We examine the limit in which the torus is shrunk onto the ring. As we cross the disc enclosed by the ring, there is a discontinuity in the normal derivative of the metric. This lack of smoothness of the metric is caused by two layers of mass spread over the disc. One is a disc of matter external to the torus and the other is an annulus of matter internal to the toroidal shell. The mass and angular momentum of the source are made up of the contributions from the toroidal shell, the disc-like surface layer, the annulus and the material interior to the toroidal shell. The disc and the annulus give us a negative mass and angular momentum contribution. The toroidal shell has positive mass and angular momentum, and the interior region gives negligible contributions in both cases. Overall, we obtain the correct mass and angular momentum. It should be noted that this source is valid for all values of the parameters "m" and "a". The forms of the contributions are as predicted by Keres[23] and Israel[25], and closely resemble the structure procured by López[30].

It should be pointed out that our disc-like source may be unstable. This is due to the fact that the masses of the disc and annulus are negative. Negative mass is repulsive and tries to break the source apart. On the other hand positive mass is attractive and tries to hold the source together. As a result any small distortion in the shape (or symmetry) of the source may cause it to break up. We have not looked at the possible problem with the stability of the source but it may be worthy of further investigation.

In section two, we re-examine Israel's line source and get a feel for the source of the Kerr geometry. In section three, we write the Kerr metric in toroidal coordinates and discuss the geometry of the background flat metric. In section four, we propose a source of the Kerr metric, making strong use of the Kerr-Schild form of the metric. The contributions to the mass and angular momentum are calculated in detail in section five. The total mass and angular momentum of the source are made up from contributions from the toroidal shell, the annulus, the disc, and the interior region. In section six, we give a short comment on the maximal analytic solution. We conclude the chapter by summing up our results.

We shall employ the following notation throughout : Latin indices i, j, k take on values 0,1,2,3 and Greek indices α, β take on values 1,2,3. For partial differentiation of scalars we use subscripts, that is, $H_a = \partial_a H$.

2.2 The Kerr Ring: Line source approximation

To obtain some insight into the problem of the source of the Kerr geometry we return to Israel's paper on line sources [28]. The structure of the Kerr ring was examined in this paper and the mass distribution of the ring was found to have a "demipole" structure. The Kerr ring takes on the appearance of a straight line

(at least locally) in the approximation that we are very close to the ring. We can snatch some clues to the structure of the source of the Kerr geometry by introducing coordinates that behave like plane polar coordinates in the vicinity of the ring.

In Boyer-Lindquist [33] coordinates $(\bar{t}, \bar{\phi}, r, \theta)$ the Kerr metric is :

$$\begin{aligned} ds^2 &= \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2 + \frac{2mra^2}{\Sigma}\sin^2\theta)\sin^2\theta\, d\bar{\phi}^2 \\ &\quad + \frac{4mar}{\Sigma}\sin^2\theta\, d\bar{\phi}\, d\bar{t} - \left(1 - \frac{2mr}{\Sigma}\right) d\bar{t}^2 \end{aligned} \tag{2.2.1}$$

where $\Delta = r^2 - 2mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. We introduce new coordinates (ρ, ψ) defined by

$$a + \rho\cos\psi = \sqrt{r^2 + a^2}\,\sin\theta \tag{2.2.2}$$

$$\rho\sin\psi = r\cos\theta\ . \tag{2.2.3}$$

We can express r and θ explicitly in terms of our new coordinates and find

$$r^2 = a\rho\cos\psi + \frac{\rho^2}{2} + a\rho\sqrt{1 + \frac{\rho}{a}\cos\psi + \frac{\rho^2}{4a^2}} \tag{2.2.4}$$

$$\cos^2\theta = -\frac{\rho}{a}\cos\psi - \frac{\rho^2}{2a^2} + \frac{\rho}{a}\sqrt{1 + \frac{\rho}{a}\cos\psi + \frac{\rho^2}{4a^2}} \tag{2.2.5}$$

In the proximity of the ring, (that is, for ρ very small), ρ and ψ become plane polar coordinates in a plane perpendicular to the ring. Thus to first order approximation in ρ we have $r^2 \approx a\rho(1+\cos\psi)$ and $\cos^2\theta \approx \frac{\rho}{a}(1-\cos\psi)$. Through the use of trigonometric identities we find

$$r \approx \sqrt{2a\rho}\,\cos\frac{\psi}{2} \tag{2.2.6}$$

$$\cos\theta \approx \sqrt{\frac{2\rho}{a}}\,\sin\frac{\psi}{2}\ . \tag{2.2.7}$$

The asymptotic form of the metric is

$$ds^2 \approx d\rho^2 + \rho^2\, d\psi^2 + a^2\, d\bar{\phi}^2 - d\bar{t}^2 + V_0(a d\bar{\phi} + d\bar{t})^2 \qquad (2.2.8)$$

where $V_0 = V_0(\rho,\psi) = \frac{2mr}{\Sigma} \approx m\sqrt{\frac{2}{a\rho}}\cos\frac{\psi}{2}$. The components of the Riemann tensor for a metric of the above form are [these components are exact for the metric given by equation (2.2.8)]

$$R_{ijkl} = 4\, l_{[i}\, V_{0|j]}[_k l_{l]} \qquad (2.2.9)$$

where the covariant differentiation is with respect to the flat metric,

$$d\eta^2 = d\rho^2 + \rho^2\, d\psi^2 + a^2\, d\bar{\phi}^2 - d\bar{t}^2 \qquad (2.2.10)$$

and $l_i = \frac{1}{\sqrt{2}}(\delta_i^{\bar{t}} + a\,\delta_i^{\bar{\phi}})$ is a null vector. The Ricci tensor is

$$R_{ij} = -(\nabla^2 V_0)\, l_i l_j \qquad (2.2.11)$$

where $\nabla^2 = \partial_{\rho\rho} + \rho^{-1}\partial_\rho + \rho^{-2}\partial_{\psi\psi}$. For the Kerr metric, V_0 satisfies $\nabla^2 V_0 = 0$, (that is, the Kerr metric is a vacuum solution of Einstein's field equations).

We are interested in finding the source of the Kerr black hole geometry. In particular, we would like to change the nature of the singularity; replacing the Kerr ring singularity with a different type of singularity that has a milder behaviour. The outline of Israel's approach is to surround the ring with a thin toroidal shell at $\rho = \rho_0$ such that the geometry exterior to the shell is given completely by the Kerr metric. Interior to the shell a gentler metric is chosen, preferably one that gives negligible mass and angular momentum contributions to the source. The interior metric is assumed to have the approximate form given by equation (2.2.8). It is noticed that the choice

$$V = m\rho_0\sqrt{\frac{\rho}{a}}\cos\frac{\psi}{2} \qquad \text{for} \qquad \rho < \rho_0 \qquad (2.2.12)$$

satisfies $\nabla^2 V = 0$. Therefore, the source for the Kerr metric seems to behave like a thin shell of matter at $\rho = \rho_0$, with a vacuum metric exterior and interior to the shell. This is not the complete story and a better picture will be presented later on.

2.3 Choice of coordinates

In the previous section, we used approximation techniques to ascertain some of the properties of the Kerr source. We shall now carry out our analysis using more refined techniques. Again, we surround the ring singularity with a toroidal tube. The toroidal tube is filled with some appropriate matter to lessen the drastic behaviour of the singularity. Our main concern will be that our source gives the correct mass and angular momentum for the Kerr black hole. By choosing an appropriate coordinate system we can greatly facilitate our calculations. Since we will be dealing with a toroidal tube, it would be advantageous to choose a coordinate system so that the equation of the tube is as simple as possible, namely toroidal coordinates. For convenience, we shall show how these are related to quasispheroidal[34] and Boyer-Lindquist[33] coordinates.

In quasispheroidal coordinates[34] (t, ϕ, r, θ) the Kerr metric is :

$$\begin{aligned} ds^2 &= \Sigma \left(d\theta^2 + \frac{dr^2}{r^2 + a^2} \right) + (r^2 + a^2) \sin^2\theta \, d\phi^2 - dt^2 \\ &\quad + \frac{2mr}{\Sigma} \left(\frac{\Sigma}{r^2 + a^2} dr + a \sin^2\theta \, d\phi + dt \right)^2 . \end{aligned} \tag{2.3.13}$$

This is related to Boyer-Lindquist[33] coordinates $(\bar{t}, \bar{\phi}, r, \theta)$ by

$$d\phi = d\bar{\phi} - \frac{2amr}{r^2 + a^2} \frac{1}{\Delta} dr \tag{2.3.14}$$

$$dt = d\bar{t} + \frac{2mr}{\Delta} dr . \tag{2.3.15}$$

To obtain toroidal coordinates (t, ϕ, σ, ψ) we make a change of variables

$$r = aN^{-1}\cos\frac{\psi}{2} \tag{2.3.16}$$

$$\sin\theta = N^{-1}\sinh\frac{\sigma}{2} \tag{2.3.17}$$

where $N = \sqrt{(\cosh\sigma - \cos\psi)/2}$, $(0 \leq \sigma \leq \infty, -\pi \leq \psi \leq \pi)$ and we find that

$$ds^2 = d\eta^2 + \frac{2m}{a}N\cos\frac{\psi}{2}(v_i dx^i)^2 \tag{2.3.18}$$

where

$$d\eta^2 = \frac{a^2}{4N^4}(d\sigma^2 + d\psi^2 + \sinh^2\sigma\, d\phi^2) - dt^2 \tag{2.3.19}$$

$$v_i dx^i = dt + \frac{a}{N^2}\sinh^2\frac{\sigma}{2}\,d\phi - \frac{a}{N^3}\cos\frac{\psi}{2}\tanh\frac{\sigma}{2}\,d\sigma - \frac{a}{2N^3}\sin\frac{\psi}{2}\,d\psi \ . \tag{2.3.20}$$

At this point, it is worth giving a short note on the nature of the Euclidean background metric, $d\eta^2$, which is just the metric of Minkowski space-time. The relationship between quasispheroidal coordinates (r, θ) and cylindrical coordinates $(\bar{\rho}, z)$ is

$$\bar{\rho} = \sqrt{r^2 + a^2}\sin\theta \qquad z = r\cos\theta \ . \tag{2.3.21}$$

We can see that $\bar{\rho}, z$ satisfy the following equations

$$\bar{\rho}^2 + (z - a\cot\psi)^2 = a^2\mathrm{cosec}^2\psi \tag{2.3.22}$$

$$(\bar{\rho} - a\coth\sigma)^2 + z^2 = a^2\mathrm{cosech}^2\sigma \ . \tag{2.3.23}$$

Therefore, in these coordinates, the surfaces $\sigma = \sigma_0$ are tori with circular cross-section ("Euclidean radius") $a\mathrm{cosech}\sigma_0$ whose central axes form circles around the z-axis with "Euclidean radius" $a\coth\sigma_0$. For any point P, we have AP/BP = $\exp\sigma$ and ψ = lower of the two angles APB, the sign being given by the sign of z, see figure 2.1.

2.4 Proposed Source

As part of the source, we construct a thin-shell of matter at $\sigma = \sigma_0$. Exterior to this shell ($\sigma < \sigma_0$) we have the Kerr metric, and on the interior ($\sigma > \sigma_0$) we pick a new metric, such that the two metrics join continuously at $\sigma = \sigma_0$. Consider a metric of the form

$$ds^2 = d\eta^2 + 2H(v_i dx^i)^2 \tag{2.4.24}$$

where $d\eta^2$ and $v_i dx^i$ are specified by (2.3.19) and (2.3.20), respectively. If $H = \frac{m}{a} N \cos\frac{\psi}{2}$ then the metric (2.4.24) is the Kerr metric. Comparing the full form of the Kerr metric (2.4.24) to the asymptotic metric (2.2.8) of the line source approximation in section two, we observe that the scalar H is analogous to the scalar V_0 of section two. Based on this analogy we forward an ansatz for the functional form of H in the interior region. Directly comparing $2H = \frac{2m}{a} N \cos\frac{\psi}{2}$ and $V_0 = m\sqrt{\frac{2}{a\rho}} \cos\frac{\psi}{2}$ we find $N \approx \sqrt{\frac{a}{2\rho}}$ (this equivalence only holds for small values of ρ). Indeed, we can derive this asymptotic identity another way. Equations (2.2.2) and (2.2.3) imply

$$\rho^2 = (\sqrt{r^2 + a^2}\sin\theta - a)^2 + r^2\cos^2\theta \tag{2.4.25}$$

$$= r^2 + a^2(1 + \sin^2\theta) - 2a\sqrt{r^2 + a^2}\sin\theta \tag{2.4.26}$$

We now substitute for r and $\sin\theta$ using equations (2.3.16) and (2.3.17), respectively and find

$$\rho^2 = \frac{a^2 e^{-\sigma}}{N^2} \tag{2.4.27}$$

For large σ, we have $N^2 \approx e^{\sigma}/4$. Therefore, re-arranging (2.4.27) we find $N \approx \sqrt{\frac{a}{2\rho}}$. The main point to note is that $N \sim \frac{1}{\sqrt{\rho}}$. We saw that the choice $V = m\rho_0\sqrt{\frac{\rho}{a}}\cos\frac{\psi}{2}$ for the interior region in the line source approximation gave a vacuum solution. This strongly suggests choosing $H \sim N_0^2/N$ in the interior region.

Formally, we propose that the source of the Kerr black hole geometry is governed by the above metric, (2.4.24), where

$$H = \begin{cases} \frac{m}{a} N \cos\frac{\psi}{2} & \text{if } \sigma < \sigma_0 \\ \frac{m}{a}\frac{N_0^2}{N}\cos\frac{\psi}{2} & \text{if } \sigma > \sigma_0 \end{cases} \quad \text{with } N_0 = \sqrt{(\cosh\sigma_0 - \cos\psi)/2} \ . \quad (2.4.28)$$

This interior ansatz for H does not give a vacuum interior solution, unlike the ansatz put forward in the line source approximation in section two. Indeed, the expression for the stress-energy tensor is quite horrendous. However, this ansatz does have a saving grace. As will be seen later, the mass of the interior region is identically zero for this choice of H. The angular momentum of the interior region is finite for a finite choice of σ_0 with the added bonus that as we take the limit σ_0 goes to infinity the angular momentum tends to zero. Indeed, we examined a whole class of ansätze for the interior form of H. We considered $H = \frac{m}{a}\frac{N_0^{n+2}}{N^{n+1}}\cos\frac{\psi}{2}$ where n is an integer, and found that $n = 0$ was the only case that gave zero for the mass of the interior region and zero for the limiting angular momentum of the interior region (we desire that theinterior metric have these two properties, thus, retaining as close a resemblance as possible to the the line source approximation). Therefore, we consider (2.4.28) to be the most likely ansatz for the interior region.

2.5 Mass and Angular Momentum of the Source

We need to show that our source gives the correct mass and angular momentum. We write Tolman's mass formula[35] in a covariant form using Gauss's theorem applied to a 3-surface embedded in a four dimensional spacetime. For an anti-symmetric tensor, F^{ij}, Gauss's theorem implies

$$\int_{\Sigma^+} F^{ij}|_j \, d\Sigma_i = \frac{1}{2}\int_{S_{out}} F^{ij} \, d\Sigma_{ij} - \frac{1}{2}\int_{S_{in}} F^{ij} \, d\Sigma_{ij} \quad (2.5.29)$$

where S_{in} and S_{out} are closed two surfaces contained in the 3-surface Σ. The region Σ_+ is subset of Σ, with inner boundary, S_{in}, and outer boundary, S_{out}. The tensors $d\Sigma_i$ and $d\Sigma_{ij}$ are essentially the volume element of Σ_+ and the surface element of S, respectively, and are given by

$$d\Sigma_i = \sqrt{-g}\,\varepsilon_{ijkl}\,\frac{\partial x^j}{\partial \xi^1}\,\frac{\partial x^k}{\partial \xi^2}\,\frac{\partial x^l}{\partial \xi^3}\,d\xi^1\,d\xi^2\,d\xi^3 \tag{2.5.30}$$

$$d\Sigma_{ij} = \sqrt{-g}\,\varepsilon_{ijkl}\,\frac{\partial x^k}{\partial \zeta^1}\,\frac{\partial x^l}{\partial \zeta^2}\,d\zeta^1\,d\zeta^2 \ . \tag{2.5.31}$$

The surface Σ and S have intrinsic coordinates $\{\xi^\alpha\}$ and $\{\zeta^A\}$, respectively. Alternately, Σ has parametric equation $x^i = x^i(\xi^1, \xi^2, \xi^3)$ and S is given by $x^i = x^i(\zeta^1, \zeta^2)$.

In our case, we take Σ to be the 3-surface $t = constant$. The integrals are performed over the surface of the source, and the interior and exterior regions. The upper surfaces of the disc spanning the ring and the annulus interior to the toroidal shell are denoted by D^+ and A^+, respectively, and the lower surfaces by D^- and A^-. The symbols T^+ and T^- indicate the outer and inner surface of the toroidal shell, respectively. The region exterior to the toroidal shell is denoted by Σ^+ whereas we use Σ^- for the interior region. These regions and surfaces are clearly indicated in figure 2.2.

The mass of the source is considered to be made up of the masses of the individual components that constitute the source. The mass of the source is the sum of the masses of the disc, annulus, toroidal shell, and the interior region. (The exterior region does not give any contributions since it is a vacuum.)

$$M_{TOTAL} = M_{DISC} + M_{ANNULUS} + M_{TORUS} + M_{INTERIOR} \ . \tag{2.5.32}$$

We think of the contribution to the total mass of the source obtained from integrating over the disc as the mass of the disc and likewise for the other components

that make up the source. This is not strictly correct since in general relativity we do not possess a local definition of energy. However, this association seems the most natural one to make. We view the angular momentum contributions in a similar manner.

The mass of the disc is

$$M_{DISC} = \frac{1}{8\pi}\int_{D^+} \xi^{i|j}\, d\Sigma_{ij} + \frac{1}{8\pi}\int_{D^-} \xi^{i|j}\, d\Sigma_{ij} \qquad (2.5.33)$$

where $\xi^i = \delta^i_t$ is the Killing vector field corresponding to invariance of the metric under time translations. (We have taken $F^{ij} = \xi^{i|j}$ in Gauss's theorem since $\xi^{i|j}$ is anti-symmetric for ξ^i a Killing vector.) The expressions for the mass of the annulus and the toroidal shell have the same form as equation (2.5.33), except that $D^\pm$ is replaced by $A^\pm$, $T^\pm$, respectively. The inner region contributes a mass

$$M_{INTERIOR} = -\frac{1}{4\pi}\int_{\Sigma^-} R^i_j\, \xi^j\, d\Sigma_i \ . \qquad (2.5.34)$$

This is a consequence of the Ricci identity, $\xi_{i|jk} - \xi_{i|kj} = \xi_l\, R^l{}_{ijk}$; when ξ^i is a Killing vector we have $\xi^{i|j}{}_{|j} = -R^i_j\, \xi^j$.

The equations for the angular momentum contributions are practically the same as for the mass contributions; however, the Killing vector that appears is $\xi^i = \delta^i_\phi$, which is associated with the invariance of the metric under rotations about its axis of symmetry. An extra factor of a half also appears. We find

$$J_{DISC} = -\frac{1}{16\pi}\int_{D^+} \xi^{i|j}\, d\Sigma_{ij} - \frac{1}{16\pi}\int_{D^-} \xi^{i|j}\, d\Sigma_{ij} \qquad (2.5.35)$$

$$J_{INTERIOR} = \frac{1}{8\pi}\int_{\Sigma^-} R^i_j\, \xi^j\, d\Sigma_i \ . \qquad (2.5.36)$$

The total angular momentum, J_{TOTAL}, is given by the analogous formula to (2.5.32).

a) Calculation of Mass Contributions

The formula for the mass of the disc, (2.5.33), reduces to

$$M_{DISC} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\sigma_0} d\sigma \,(\sqrt{-g}\,\xi^{\psi|t})_{D^+} + \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{\sigma_0}^{0} d\sigma \,(\sqrt{-g}\,\xi^{\psi|t})_{D^-} \quad . \tag{2.5.37}$$

For Kerr-Schild metrics (see Appendix A), the determinant of the metric is identical to the determinant of the Minkowski metric, that is, $g = \eta$. Therefore, we have

$$\sqrt{-g} = \frac{a^3}{4N^6}\cosh\frac{\sigma}{2}\sinh\frac{\sigma}{2} \quad . \tag{2.5.38}$$

In particular on the upper and lower surfaces of the disc

$$\sqrt{-g} = \frac{a^3}{4}\frac{\sinh\frac{\sigma}{2}}{\cosh^5\frac{\sigma}{2}} \quad . \tag{2.5.39}$$

The covariant derivative of the Killing vector field can be calculated using equation (A.15) since the Killing vector is of the desired form. We find

$$(\xi^{\psi|t})_{D^\pm} = (\eta^{\psi\psi}\,H_\psi)_{\psi=\pm\pi} = \mp\frac{2m}{a^3}\cosh^5\frac{\sigma}{2} \quad . \tag{2.5.40}$$

Therefore, the mass of the disc is

$$M_{DISC} = -\frac{m}{2}\int_0^{\sigma_0} d\sigma \,\sinh\frac{\sigma}{2} \quad . \tag{2.5.41}$$

The result after performing the integral is the exact formula

$$M_{DISC} = m - m\cosh\frac{\sigma_0}{2} \quad . \tag{2.5.42}$$

The calculation of the mass of the annulus is similar to that of the disc, the differences are that the range of integration for σ changes and that the interior choice of the scalar H must be used, [see equation (2.4.28)],

$$M_{ANNULUS} = \frac{1}{2}\int_{\sigma_0}^{\infty} d\sigma \,\{(\sqrt{-g}\,\xi^{\psi|t})_{A^+} - (\sqrt{-g}\,\xi^{\psi|t})_{A^-}\} \tag{2.5.43}$$

where

$$(\xi^{\psi|t})_{A\pm} = (\eta^{\psi\psi} H_\psi)_{A\pm} = \mp \frac{2m}{a^3} \cosh^3 \frac{\sigma}{2} \cosh^2 \frac{\sigma_0}{2} \quad . \tag{2.5.44}$$

Therefore, we have

$$M_{ANNULUS} = -\frac{m}{2} \cosh^2 \frac{\sigma_0}{2} \int_{\sigma_0}^{\infty} d\sigma \, \frac{\sinh \frac{\sigma}{2}}{\cosh^2 \frac{\sigma}{2}} = m \cosh^2 \frac{\sigma_0}{2} \left[\frac{1}{\cosh \frac{\sigma}{2}}\right]\Big|_{\sigma_0}^{\infty} \quad . \tag{2.5.45}$$

The mass of the annulus is

$$M_{ANNULUS} = -m \cosh \frac{\sigma_0}{2} \tag{2.5.46}$$

which again is an exact formula.

The mass of the toroidal shell is

$$M_{TORUS} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-\pi}^{\pi} d\phi \, \{(\sqrt{-g}\, \xi^{t|\sigma})_{T+} - (\sqrt{-g}\, \xi^{t|\sigma})_{T-}\} \tag{2.5.47}$$

where

$$(\xi^{t|\sigma})_{T\pm} = -(\eta^{\sigma\sigma} H_\sigma)_{T\pm} = \mp \frac{2m}{a^3} N_0^3 \sinh \frac{\sigma_0}{2} \cosh \frac{\sigma_0}{2} \cos \frac{\psi}{2} \quad . \tag{2.5.48}$$

This implies that

$$M_{TORUS} = \frac{m}{2} \sinh^2 \frac{\sigma_0}{2} \cosh^2 \frac{\sigma_0}{2} \int_{-\pi}^{\pi} d\psi \, \frac{\cos \frac{\psi}{2}}{N_0^3} \quad . \tag{2.5.49}$$

Looking this integral up in our tables (C.11) the mass of the toroidal shell is

$$M_{TORUS} = 2m \cosh \frac{\sigma_0}{2} \quad . \tag{2.5.50}$$

The components of the Ricci tensor (or in other words the stress-energy tensor) must be known before the contribution to the mass can be calculated since

$$M_{INTERIOR} = -\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-\pi}^{\pi} d\psi \int_{\infty}^{\sigma_0} d\sigma \, \sqrt{-g}\, R_t^t \tag{2.5.51}$$

where equation (2.5.34) has been used. The relevant component of the Ricci tensor can be calculated using equation (B.17) which tells us that

$$-\sqrt{-g}\,R^t_t = \sqrt{-g}\,\Box H \doteq \partial_i(\sqrt{-g}\eta^{ij}\,\partial_j H) \tag{2.5.52}$$

$$= a[\partial_\sigma(\frac{\sinh\frac{\sigma}{2}\cosh\frac{\sigma}{2}}{N^2}H_\sigma) + \partial_\psi(\frac{\sinh\frac{\sigma}{2}\cosh\frac{\sigma}{2}}{N^2}H_\psi)] \tag{2.5.53}$$

At a glance we see that $\sqrt{-g}\,R^t_t = \mathcal{O}(e^{-\sigma/2})$. Therefore equation (2.5.51) implies $M_{INTERIOR} = \mathcal{O}(e^{-\sigma_0/2})$ and for σ_0 very large this gives $M_{INTERIOR} \approx 0$. Actually a peculiar state of affairs exists for the particular ansatz for H that we made [see (2.4.28) for the ansatz]. We can show that $M_{INTERIOR}$ is identically zero for this choice of H even though R_{ij} is not equal to zero. (The interior solution is not a vacuum metric; nevertheless, the mass of the interior region is zero.) To calculate $M_{INTERIOR}$, explicitly substitute (2.5.53) into (2.5.51) to obtain

$$M_{INTERIOR} = \frac{a}{2}\int_{-\pi}^{\pi}d\psi\int_{\infty}^{\sigma_0}d\sigma\,[\partial_\sigma(\frac{\sinh\frac{\sigma}{2}\cosh\frac{\sigma}{2}}{N^2}H_\sigma) + \partial_\psi(\frac{\sinh\frac{\sigma}{2}\cosh\frac{\sigma}{2}}{N^2}H_\psi)] \tag{2.5.54}$$

$$= \frac{a}{2}\int_{-\pi}^{\pi}d\psi\,[\frac{\sinh\frac{\sigma}{2}\cosh\frac{\sigma}{2}}{N^2}H_\sigma]\Big|_{\infty}^{\sigma_0} + \frac{a}{2}\int_{-\infty}^{\sigma_0}d\sigma\,[\frac{\sinh\frac{\sigma}{2}\cosh\frac{\sigma}{2}}{N^2}H_\psi]\Big|_{-\pi}^{\pi}\,. \tag{2.5.55}$$

Now for the specific H from (2.4.28) we find

$$M_{INTERIOR} = \frac{m}{4}\sinh^2\frac{\sigma_0}{2}\cosh^2\frac{\sigma_0}{2}\int_{-\pi}^{\pi}d\psi\,\frac{\cos\frac{\psi}{2}}{N_0^3} + \frac{m}{2}\cosh^2\frac{\sigma_0}{2}\int_{\infty}^{\sigma_0}d\sigma\frac{\sinh\frac{\sigma}{2}}{\cosh^2\frac{\sigma}{2}}\,. \tag{2.5.56}$$

Using appendix C, formula (C.11), we have the result that

$$M_{INTERIOR} = 0\,. \tag{2.5.57}$$

It should be remarked that this result is an exact result and is independent of the position of the toroidal shell. The result holds for any value of σ_0 that is chosen and for our specific choice for H.

b) Calculation of Angular Momentum Contributions

The calculations for angular momentum are similar to those for mass but require a little more work. The Killing vector field that appears in the equations that follow is $\xi^i = \delta^i_\phi$, which corresponds to invariance of the metric under rotations about the axis of the black hole. Equation (2.5.35) implies that the angular momentum of the disc is

$$J_{DISC} = -\frac{1}{8\pi}\int_0^{2\pi} d\phi \int_0^{\sigma_0} d\sigma \,\{(\sqrt{-g}\,\xi^{\psi|t})_{D^+} - (\sqrt{-g}\,\xi^{\psi|t})_{D^-}\} \quad . \tag{2.5.58}$$

For the Killing vector $\xi^i = \delta^i_\phi$, equation (A.15) implies

$$(\xi^{\psi|t})_{D^\pm} = (\eta^{\psi\psi}\,\eta_{\phi\phi}\,v^\phi{}_{,\psi}\,H + v_\phi\,\eta^{\psi\psi}\,H_\psi)_{D^\pm} \tag{2.5.59}$$

$$= \mp\frac{2m}{a^2}\sinh^2\frac{\sigma}{2}\cosh^3\frac{\sigma}{2} \quad . \tag{2.5.60}$$

Therefore, we have

$$J_{DISC} = \frac{ma}{4}\int_0^{\sigma_0} d\sigma \,\sinh\frac{\sigma}{2}(1 - \frac{1}{\cosh^2\frac{\sigma}{2}}) = \frac{ma}{2}\,[\cosh\frac{\sigma}{2} + \frac{1}{\cosh\frac{\sigma}{2}}]\,|_0^{\sigma_0} \quad . \tag{2.5.61}$$

The angular momentum of the disc is

$$J_{DISC} = -ma + \frac{ma}{2}\cosh\frac{\sigma_0}{2} + \frac{ma}{2}\frac{1}{\cosh\frac{\sigma_0}{2}} \quad . \tag{2.5.62}$$

The angular momentum associated with the annulus is given by

$$J_{ANNULUS} = -\frac{1}{4}\int_0^{\sigma_0} d\sigma \,\{(\sqrt{-g}\,\xi^{\psi|t})_{A^+} - (\sqrt{-g}\,\xi^{\psi|t})_{A^-}\} \tag{2.5.63}$$

where

$$(\xi^{\psi|t})_{A^{\pm}} = \mp\frac{2m}{a^2}\cosh^2\frac{\sigma_0}{2}\sinh^2\frac{\sigma}{2}\cosh\frac{\sigma}{2} \quad . \tag{2.5.64}$$

We find that

$$J_{ANNULUS} = -\frac{ma}{4}\cosh^2\frac{\sigma_0}{2}\int_{\sigma_0}^{\infty} d\sigma \frac{\sinh^3\frac{\sigma}{2}}{\cosh^4\frac{\sigma}{2}} \tag{2.5.65}$$

$$= -\frac{ma}{2}\cosh^2\frac{\sigma}{2}_0[-\frac{1}{3}\frac{1}{\cosh^3\frac{\sigma}{2}} + \frac{1}{\cosh\frac{\sigma}{2}}]|_{\sigma_0}^{\infty} \quad . \tag{2.5.66}$$

The angular momentum of the annulus is

$$J_{ANNULUS} = \frac{ma}{2}\cosh\frac{\sigma_0}{2} - \frac{ma}{6}\frac{1}{\cosh\frac{\sigma_0}{2}} \quad . \tag{2.5.67}$$

We find that the angular momentum of the toroidal shell is given by

$$J_{TORUS} = -\frac{1}{4}\int_{\pi}^{-\pi} d\psi \{(\sqrt{-g}\,\xi^{t|\sigma})_{T^+} - (\sqrt{-g}\,\xi^{t|\sigma})_{T^-}\} \tag{2.5.68}$$

where

$$(\xi^{t|\sigma})_{T^{\pm}} = -(\eta^{\sigma\sigma}\,\eta_{\phi\phi}\,v^{\phi}{}_{,\sigma}\,H + v_{\phi}\,\eta^{\sigma\sigma}\,H_{\sigma})_{T^{\pm}} \quad . \tag{2.5.69}$$

Equation (2.5.68) reduces to the integral

$$J_{TORUS} = -\frac{ma}{4}\sinh^4\frac{\sigma_0}{2}\cosh^2\frac{\sigma_0}{2}\int_{-\pi}^{\pi} d\psi \frac{\cos\frac{\psi}{2}}{N_0^5} \quad . \tag{2.5.70}$$

Looking this integral up in appendix C, (C.12), we find that the angular momentum of the toroidal shell is

$$J_{TORUS} = -ma\cosh\frac{\sigma_0}{2} + \frac{ma}{3}\frac{1}{\cosh\frac{\sigma_0}{2}} \quad . \tag{2.5.71}$$

The angular momentum of the interior region is given by

$$J_{INTERIOR} = \frac{1}{8\pi}\int_0^{2\pi} d\phi \int_{-\pi}^{\pi} d\psi \int_{\infty}^{\sigma_0} d\sigma \sqrt{-g}\,R^t_{\phi} \quad . \tag{2.5.72}$$

But we know from Appendix B, (B.17), that

$$-\sqrt{-g}\,R^t_\phi = \sqrt{-g}(C^*_\phi + v_\phi \Box H) = \mathcal{O}(e^{-\sigma/2}) \tag{2.5.73}$$

so in the limit σ_0 goes to infinity the value of (2.5.72) is zero. The angular momentum of the interior region is

$$J_{INTERIOR} = \mathcal{O}(e^{-\sigma_0/2}) \ . \tag{2.5.74}$$

Let us now calculate the exact expression for $J_{INTERIOR}$. We use (B.17) to find an expression for R^t_ϕ. Using equations (B.3), (B.12), and the specific form of H for the interior region [see equation (2.4.28)], we eventually find that

$$\sqrt{-g}\,R^t_\phi = \frac{ma}{2}\frac{\sinh^3\frac{\sigma}{2}\cosh\frac{\sigma}{2}}{N^7}\cos\frac{\psi}{2}[\cosh\frac{\sigma_0}{2} - 4\cos^2\frac{\psi}{2}\cosh^2\frac{\sigma}{2} + 3\cosh^2\frac{\sigma}{2}] \ . \tag{2.5.75}$$

First, we shall integrate over the ψ variable. With the help of our table of integrals, (C.13) and (C.10), and a little algebra we find

$$\int_{-\pi}^{\pi} d\psi\,\sqrt{-g}\,R^t_\phi = 2ma(\cosh^2\frac{\sigma_0}{2}P_1 + P_2) \ . \tag{2.5.76}$$

The functions P_1 and P_2 are given by

$$P_1 = \frac{1}{\sinh^3\frac{\sigma}{2}}[1 - \frac{2}{3}\frac{1}{\cosh^2\frac{\sigma}{2}} + \frac{1}{5}\frac{1}{\cosh^4\frac{\sigma}{2}}] \tag{2.5.77}$$

$$P_2 = \frac{1}{\sinh^3\frac{\sigma}{2}}[\frac{1}{3}\cosh^2\frac{\sigma}{2} - \frac{22}{15} + \frac{3}{5}\frac{1}{\cosh^2\frac{\sigma}{2}}] \ . \tag{2.5.78}$$

Making the change of variable $x = \sinh\frac{\sigma}{2}$ we have

$$\int_{\infty}^{\sigma_0} d\sigma P_1 = 2\int_{\infty}^{\cosh\frac{\sigma_0}{2}} \frac{dx}{(x^2-1)^2}[1 - \frac{2}{3}\frac{1}{x^2} + \frac{1}{5}\frac{1}{x^4}] \tag{2.5.79}$$

$$\int_{\infty}^{\sigma_0} d\sigma P_2 = 2\int_{\infty}^{\cosh\frac{\sigma_0}{2}} \frac{dx}{(x^2-1)^2}[\frac{2}{3}x^2 - \frac{22}{15} + \frac{3}{5}\frac{1}{x^2}] \ . \tag{2.5.80}$$

The above integrals can be worked out using the iteration formulae (C.15), (C.16), and (C.17). We find that

$$\int_{\infty}^{\sigma_0} d\sigma P_1 = 2[\frac{1}{15}\frac{1}{x^3(x^2-1)} - \frac{1}{3}\frac{1}{x(x^2-1)}]\Big|_{\infty}^{\cosh\frac{\sigma_0}{2}} \tag{2.5.81}$$

$$\int_{\infty}^{\sigma_0} d\sigma P_2 = 2[-\frac{1}{3}\frac{x}{x^2-1} + \frac{3}{5}\frac{1}{x(x^2-1)}]\Big|_{\infty}^{\cosh\frac{\sigma_0}{2}} . \tag{2.5.82}$$

Evaluating (2.5.81) and (2.5.82), and inserting the results into (2.5.76) we have

$$\int_{\infty}^{\sigma_0} d\sigma \int_{-\pi}^{\pi} d\psi \sqrt{-g}\, R^t_{\phi} = -\frac{8ma}{3}\frac{1}{\cosh\frac{\sigma_0}{2}} . \tag{2.5.83}$$

Equation (2.5.72) then implies that the angular momentum of the interior region is

$$J_{INTERIOR} = -\frac{2ma}{3}\frac{1}{\cosh\frac{\sigma_0}{2}} . \tag{2.5.84}$$

Again, we emphasise the fact that this is an exact expression for the angular momentum of the interior region.

2.6 Maximal Analytic Solution

It is possible to extend the Kerr metric to negative values of r. As the equatorial disc is approached from either above or below, r tends to zero through positive values, but the gradient of r does not disappear. A maximal analytic continuation can be obtained by continuing r to negative values through the interior of the disc. Starting with two coordinate charts, one where r is positive and the other where r is negative, we identify points on the top of the disc in the space where r is positive with points on the bottom side of the disc in the space where r is negative. We do likewise for the top of the disc where r is negative and the bottom of the disc where r is positive. This identification is done in such a way that the

gradient of r is continuous across the disc. For large values of negative r the space is asymptotically flat. For negative values of r the ring singularity is a naked singularity of negative mass.

For the maximal analytic Kerr geometry, the source would only consist of a torus. We no longer have a disc and annulus because the normal is then continuous across $\psi = -\pm\pi$. The only contributions to the mass and angular momentum come from the torus. For the mass, we have

$$M_{TORUS} = 2\,|m|\cosh\frac{\sigma_0}{2} + (-|m|)\cosh\frac{\sigma_0}{2} \quad . \tag{2.6.85}$$

$M_{TOTAL} = 0$, which is correct since trivially, $|m| + (-|m|) = 0$. We get a similar result for the angular momentum since "a" does not change sign from $r > 0$ to $r \leq 0$. Taub in his application of Gauss's theorem neglected the fact that the space must be simply connected. He should have also performed an integral over the surface $r = -\infty$. If he had done so he would not have obtained any contradictions.

2.7 Summary

A source for the Kerr black hole geometry was presented that consists of a toroidal shell that encloses the ring singularity, a disc that spans the equatorial plane, and an annulus of matter interior to the toroidal shell. The torus was also filled with some matter that gives negligible contributions to the mass and angular momentum of the source. We recall that the individual mass contributions are given by the exact expressions

Mass contributions to the source		
M_{DISC}	$=$	$m - m\cosh\frac{\sigma_0}{2}$
$M_{ANNULUS}$	$=$	$-m\cosh\frac{\sigma_0}{2}$
M_{TORUS}	$=$	$2m\cosh\frac{\sigma_0}{2}$
$M_{INTERIOR}$	$=$	0

We can now employ equation (2.5.32) to obtain the total mass of the source:

$$M_{TOTAL} = m \tag{2.7.86}$$

demonstrating that our source does indeed give the correct mass for the black hole.

The individual angular momentum contributions are

Angular momentum contributions to the source		
J_{DISC}	$=$	$-ma + \frac{ma}{2}\cosh\frac{\sigma_0}{2} + \frac{ma}{2}\frac{1}{\cosh\frac{\sigma_0}{2}}$
$J_{ANNULUS}$	$=$	$\frac{ma}{2}\cosh\frac{\sigma_0}{2} - \frac{ma}{6}\frac{1}{\cosh\frac{\sigma_0}{2}}$
J_{TORUS}	$=$	$-ma\cosh\frac{\sigma_0}{2} + \frac{ma}{3}\frac{1}{\cosh\frac{\sigma_0}{2}}$
$J_{INTERIOR}$	$=$	$-\frac{2ma}{3}\frac{1}{\cosh\frac{\sigma_0}{2}}$

All the above expressions are again exact. If they are added together, we imme-

diately see that the total angular momentum of the source is

$$J_{TOTAL} = -ma \tag{2.7.87}$$

which is the value we expect for a rotating black hole.

So far all the results that we have stated are precise results. We are also interested in what happens in the limit that the toroidal shell is shrunk onto the ring. An immediate consequence is that the angular momentum of the interior region is zero in this limit. However, the vast majority of the contributions to the mass and angular momentum actually become infinite in this limit. Shrinking the toroidal shell onto the ring is equivalent to taking the limit σ_0 tends to infinity. The individual mass contributions take the following form

$$M_{DISC} = m - \frac{m}{2} \lim_{\sigma \to \infty} e^{\sigma/2} \tag{2.7.88}$$

$$M_{ANNULUS} = -\frac{m}{2} \lim_{\sigma \to \infty} e^{\sigma/2} \tag{2.7.89}$$

$$M_{TORUS} = m \lim_{\sigma \to \infty} e^{\sigma/2} \tag{2.7.90}$$

$$M_{INTERIOR} = 0 \tag{2.7.91}$$

and the individual angular momentum contributions become

$$J_{DISC} = -ma + \frac{ma}{4} \lim_{\sigma \to \infty} e^{\sigma/2} \tag{2.7.92}$$

$$J_{ANNULUS} = \frac{ma}{4} \lim_{\sigma \to \infty} e^{\sigma/2} \tag{2.7.93}$$

$$J_{TORUS} = -\frac{ma}{2} \lim_{\sigma \to \infty} e^{\sigma/2} \tag{2.7.94}$$

$$J_{INTERIOR} = 0 \ . \tag{2.7.95}$$

We observe that several of the individual contributions become infinite in this limit. However, all these terms cancel so that the source still has the correct

mass and angular momentum. The source then consists of a disc spanning the equatorial plan and a ring of matter. The ring has an infinite positive mass and the disc has an infinite negative mass. These two masses combine to give a finite value for the mass of the source. A similar cancellation occurs for the angular momentum. The ring of matter is made up from a shrunken toroidal shell containing an annulus of matter. The annulus and the toroidal shell both get squeezed onto the ring.

Our results are similar to the original findings of Israel[25] and the later results of López[30]. Thus, our exact results seem to indicate that Janis and Newman[22] were correct in their original assertion about the source of the Kerr metric; the source of the Kerr metric is a disc bounded by a ring of rotating matter.

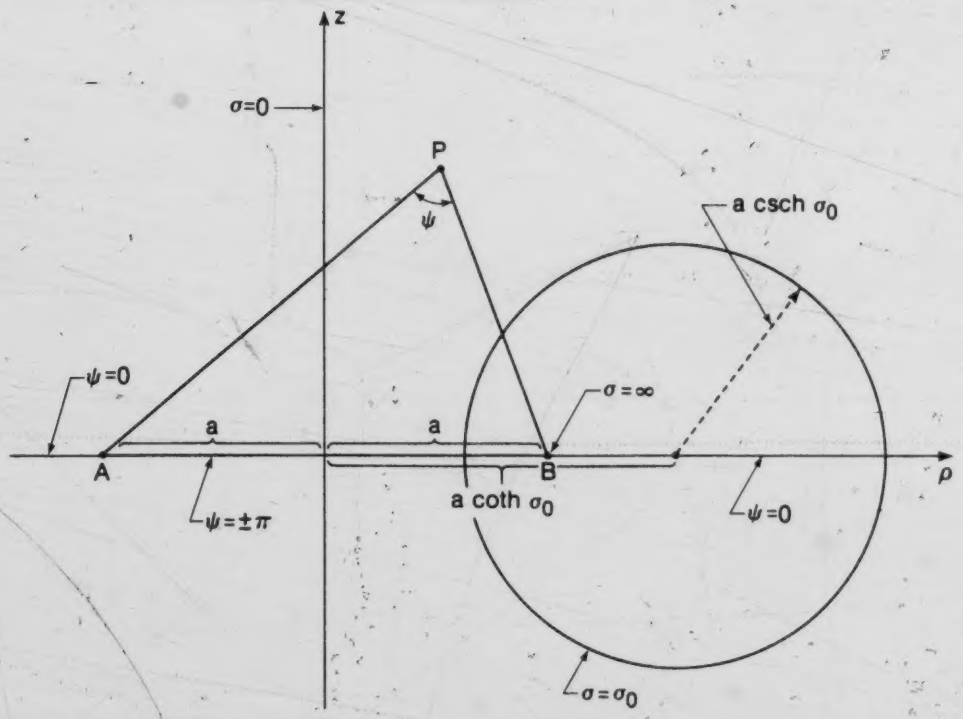

Figure 2.1: Illustrates the relationship between toroidal and cylindrical coordinates. A cross-section at constant ϕ is shown. The toroidal shell is given by $\sigma = constant$. The Kerr ring is situated at $\sigma = \infty$. The upper half of the line segment AB is given by $\psi = +\pi$ and the lower half by $\psi = -\pi$. For an arbitrary point P, the coordinate ψ is given by the lower of the two angles $\angle APB$.

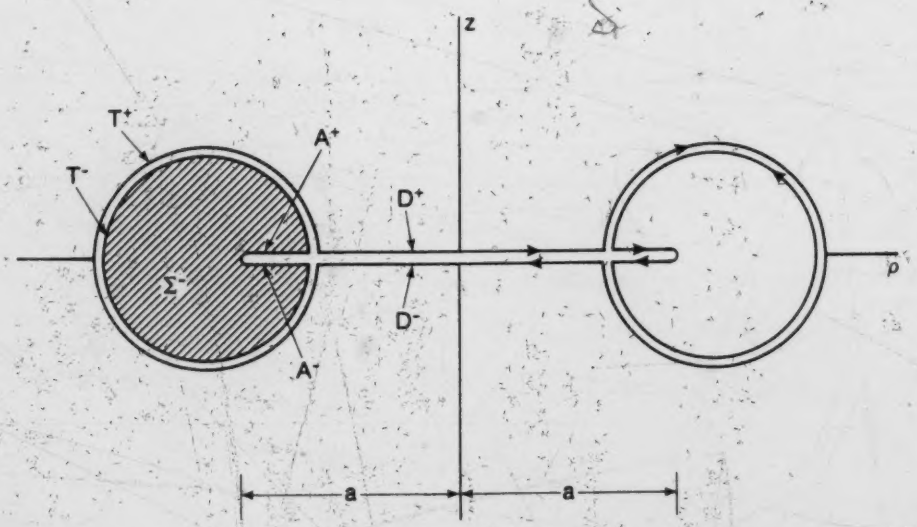

Figure 2.2: Source of Kerr metric. Clearly illustrating the upper and lower surfaces of the disc and annulus. We can also see the inner and outer surfaces of the toroidal shell. The arrows indicate the directions in which we have to performed our integrations in order to comply with Gauss's theorem.

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CHAPTER THREE

A TOROIDAL SOURCE FOR THE KERR-NEWMAN BLACK HOLE GEOMETRY

3.1 Introduction

The Kerr-Newman metric is the charged analogue of the Kerr metric[1]. Newman *et al.*[2] first discovered the metric by performing a complex coordinate transformation on the Reissner-Nordström metric[3,4]. (The Reissner-Nordström metric describes the gravitational field outside a non rotating charged black hole.) The Kerr-Newman metric was initially interpreted by Newman *et al.* as representing "the gravitational and electromagnetic fields of a ring of mass and charge rotating about its axis of symmetry." They were aware that there existed a problem with their interpretation and indicated that source may also include a disc of material whose rim was the Kerr ring.

Several efforts have been made since to find the source of the Kerr-Newman geometry. Most of the early endeavors concentrated on examining possible disc-like sources, the most notable being the work of Israel[5] and López[6].

While the Kerr-Newman metric is continuous across the disc $r = 0$, the normal to the disc is discontinuous. This lack of smoothness was considered by Israel to be caused by a layer of mass and charge spread over the disc. The general characteristics of the disc were found using the theory of surface layers and a "subtraction argument" was used to infer the nature of the ring singularity. The ring singularity was deduced to contribute infinite positive mass and angular momentum, and to carry infinite positive charge.

López[6,7] extended his earlier work on the source of the Kerr metric to examine the source of the Kerr-Newman geometry. He took both the metric and the electromagnetic potential to be distributions defined at all points in spacetime. He obtained a stress-energy tensor that had its support throughout the disc and along the ring. His stress-energy tensor lead to the correct values for the mass, angular momentum, and charge of the Kerr-Newman geometry.

Several authors have forwarded the idea that the spinning electron may be classically viewed as a massive charged source of the Kerr-Newman [8,5,9] field. The Kerr-Newman field has a gyromagnetic ratio $\mu/ma = e/m$ which coincides with the ratio for the Dirac electron. The physical properties of the electron dictate that the Kerr-Newman parameter must be specified as follows: $m \simeq 10^{-22}, ma = \frac{1}{2}$ and $e^2 = \frac{1}{137}$ in units $G = 1, \hbar = 1, c = 1$. The radius $a \simeq 10^{22}$ is the same length scale as the Compton wavelength of the electron.

López assumed that there was a thin rigidly rotating shell at $r = e^2/2m$. The Kerr-Newman metric can be matched at this surface to a Minkowski vacuum interior. However, the electromagnetic potentials are not constant on this surface. Therefore, the surface not only contains charge but also electric dipoles. The electron is modeled as a bubble of flat spacetime immersed in the Kerr-Newman geometry.

In this chapter, we extend the work of the previous chapter on the source of the Kerr black hole geometry. We propose a toroidal source for the Kerr-Newman black hole geometry. We find that the source is much the same as before except now with the addition of electric charge. Our source consists of a toroidal tube surrounding the ring, a disc of charged matter external to the toroidal shell, and an annulus of matter internal to the toroidal shell. The toroidal shell is filled

with some material whose mass and angular momentum are almost entirely of electromagnetic origin. The mass and angular momentum of the source are made up form contributions from the disc-like surface layer, the annulus of matter, the toroidal shell, and the matter internal to the toroidal shell. The region external to the source is no longer a vacuum; an electromagnetic field permeates the exterior region. The electromagnetic energy density of this field contributes to the total mass of the source and the Poynting vector of the field contributes to the angular momentum of the source. The electromagnetic field gives positive contributions on both accounts and once the electromagnetic field is included in the calculations we find that our source has the correct mass and angular momentum. The masses of the disc and the annulus are negative, and the mass of the toroidal shell depends on the magnitude of the charge of the source. The same findings hold for the angular momentum except now that the angular momenta of the disc and annulus are positive. The electric charge of the source is a little bit trickier to comment upon since we do not have a handle on the interior electromagnetic field. However, we can show by purely considering properties of the electromagnetic field on the surface of the source that the source has the correct charge.

In section two of this chapter, we express the Kerr-Newman metric in toroidal coordinates and discuss a possible ansatz for the interior metric. We calculate the mass, angular momentum, and charge contributions of the various parts of the source in section three. We end the chapter with a summary of our results.

3.2 The metric

The Kerr-Newman metric is essentially the same as the Kerr metric but with the Kerr mass parameter replaced by the effective mass. In quasispheroidal coordinates[10] (t, ϕ, r, θ) the effective mass is

$$m_{\text{eff}} = m - \frac{e^2}{2r} \tag{3.2.1}$$

and the Kerr-Newman line-element looks like

$$\begin{aligned} ds^2 &= \Sigma \left(d\theta^2 + \frac{dr^2}{r^2 + a^2} \right) + (r^2 + a^2) \sin^2\theta \, d\phi^2 - dt^2 \\ &\quad + \frac{2mr - e^2}{\Sigma} \left(\frac{\Sigma}{r^2 + a^2} \, dr + a \sin^2\theta \, d\phi + dt \right)^2 \end{aligned} \tag{3.2.2}$$

where $\Sigma = r^2 + a^2 \cos^2\theta$. As in the previous chapter, our search for the source of the Kerr-Newman geometry is greatly helped by expressing the metric in toroidal coordinates (t, ϕ, σ, ψ). We remind ourselves that toroidal coordinates are related to quasispheroidal coordinates via

$$r = aN^{-1} \cos\frac{\psi}{2} \tag{3.2.3}$$

$$\sin\theta = N^{-1} \sinh\frac{\sigma}{2} \tag{3.2.4}$$

where $N = \sqrt{\sinh^2\frac{\sigma}{2} + \sin^2\frac{\psi}{2}} \equiv \sqrt{\cosh^2\frac{\sigma}{2} - \cos^2\frac{\psi}{2}}$. The metric (3.2.2) takes the usual form

$$ds^2 = d\eta^2 + 2H \, (v_i dx^i)^2 \tag{3.2.5}$$

where $d\eta^2$ is a flat metric expressed in toroidal coordinates and v_i is a null vector. We recollect from chapter two that

$$d\eta^2 = \frac{a^2}{4N^4} (d\sigma^2 + d\psi^2 + \sinh^2\sigma \, d\phi^2) - dt^2 \tag{3.2.6}$$

$$\begin{aligned} v_i dx^i &= dt + \frac{a}{N^2} \sinh^2\frac{\sigma}{2} \, d\phi - \frac{a}{N^3} \cos\frac{\psi}{2} \tanh\frac{\sigma}{2} \, d\sigma \\ &\quad - \frac{a}{2N^3} \sin\frac{\psi}{2} \, d\psi \ . \end{aligned} \tag{3.2.7}$$

The scalar $H\,(=\frac{2mr-e^2}{\Sigma})$ expressed in toroidal coordinates is

$$H = \frac{m}{a} N \cos\frac{\psi}{2} - \frac{e^2}{2a^2} N^2 \quad . \tag{3.2.8}$$

Our aim is to construct a source for the Kerr-Newman black hole geometry. We pursue the same approach we implemented for the source of the Kerr black hole geometry, (see chapter two for details). As a large fragment of the source we fabricated a thin shell of matter at $\sigma = \sigma_0$. The geometry of the spacetime exterior to this shell is completely specified by the Kerr-Newman metric. We have the task of postulating some reasonable interior metric that will give rise to the Kerr-Newman exterior geometry. The scalar H is composed of two parts as can clearly be perceived from equation (3.2.8): (i) the mass term which gives the Kerr solution, and (ii) the "e^2" term which encodes the information about how the electro-magnetic field affects the geometry. A reasonable assumption to make is that our choice for an interior metric must reduce to our previous choice for the Kerr source. A natural selection for the mass term is to chose the value of H we picked for the interior Kerr solution, [see equation (2.4.28)]. We recall that this choice was motivated by having an interior solution that behaved like a vacuum solution in the sense of having zero mass and angular momentum. We do not possess any such guiding principle for the e^2 term since this term produces an electro-magnetic field. However, we do know that the metric must be continuous at $\sigma = \sigma_0$. We put forward the following ansatz for the interior metric: the interior metric is given by equation (3.2.5) where the scalar H has the form

$$H = \frac{m}{a}\frac{N_0^2}{N} \cos\frac{\psi}{2} - \frac{e^2}{2a^2}\frac{N_0^{n+2}}{N^n} \tag{3.2.9}$$

where n is a real number and $N_0 = N|_{\sigma=\sigma_0}$. We shall leave n unspecified for now but will later determine the most desirable value.

3.3 Mass, Angular Momentum, and Electric Charge of the Source

We must show that our source gives the correct mass, angular momentum, and charge for the Kerr-Newman black hole. The source consists of a toroidal shell that encloses the ring singularity, a disc that spans the equatorial plane, an annulus of matter interior to the torus, and some matter inside the toroidal shell. The mass of the source can be considered as the sum of the individual masses of the constituent parts that make up the source. We view the total angular momentum and charge in a similar light. The calculations for the mass and angular momentum will be carried out along the lines of the previous chapter. To calculate the electric charge we make use of Gauss's law.

The total mass of the source is made up from contributions from the various parts so that

$$M_{TOTAL} = M_{EXTERIOR} + M_{INTERIOR} + M_{ANNULUS} + M_{DISC} + M_{TORUS} \ . \tag{3.3.10}$$

The Kerr-Newman solution is an electro-vac solution, and as a result the presence of an electromagnetic field exterior to the source contributes $M_{EXTERIOR}$ to the mass of the source (this is the electrostatic energy density).

We can calculate the individual mass contributions by using the following formulae:

$$M_{EXTERIOR} = -\frac{1}{4\pi}\int_{\Sigma^+} R^t_t \sqrt{-g}\, d\phi\, d\psi\, d\sigma \tag{3.3.11}$$

$$M_{DISC} = \frac{1}{8\pi}\int_{D^+} \xi^{i|j}\, d\Sigma_{ij} + \frac{1}{8\pi}\int_{D_-} \xi^{i|j}\, d\Sigma_{ij} \tag{3.3.12}$$

where $\xi^i = \delta^i_t$ is the time-like Killing vector field associated with the invariance of the metric under time translations, D^+ and D^- are upper and lower surfaces

of the disc respectively, and $d\Sigma_{ij}$ is the surface element of the disc. The formulae for M_{TORUS} and $M_{ANNULUS}$ are similar to (3.3.12), but the integrals are over the surfaces of the toroidal shell and the annulus, respectively. The formula for $M_{INTERIOR}$ is the same as (3.3.11), except that we integrate over the interior volume, Σ^-; see figure 2.2 in the preceding chapter.

We can calculate the total angular momentum in a similar fashion. We have

$$J_{TOTAL} = J_{EXTERIOR} + J_{INTERIOR} + J_{DISC} + J_{ANNULUS} + J_{TORUS} \quad (3.3.13)$$

where

$$J_{EXTERIOR} = \frac{1}{8\pi} \int_{\Sigma^+} R^i_\phi \sqrt{-g}\, d\phi\, d\psi\, d\sigma_i \quad (3.3.14)$$

$$J_{DISC} = -\frac{1}{16\pi} \int_{D^+} \xi^{i|j} d\Sigma_{ij} - \frac{1}{16\pi} \int_{D^-} \xi^{i|j} d\Sigma_{ij} \ . \quad (3.3.15)$$

But now $\xi^i = \delta^i_\phi$ is the space-like Killing vector field generated by axial symmetry. Similar formulae exist for the angular momentum of the annulus, toroidal shell, and the interior region.

We immediately notice that the formulae for calculating the mass and angular momentum contributions (3.3.11), (3.3.12), (3.3.14), and (3.3.15) are all linear in the scalar H and its derivatives. Therefore, we need only calculate those contributions that contain e^2 explicitly since the others will be exactly the same as the contributions to the Kerr source which we have already calculated in the previous chapter.

We find that we can express the total charge of the source entirely in terms of quantities calculated on the exterior surface of the source and we do not need to have the specific form of the electromagnetic potential in the interior. The electromagnetic field can be expressed as a differential 2-form,

$$F = dA = \frac{1}{2} F_{ij}\, dx^i \wedge dx^j \ . \quad (3.3.16)$$

where $A = A_i\,dx^i$ is the electromagnetic potential expressed as a differential 1-form. The electromagnetic field satisfies Maxwell's equations; $F_{[ij|k]} = 0$ and $F^{ij}\,|_j = 4\pi\,J^i$ where J^i is the relativistic current density. We can use the latter of Maxwell's equations to obtain an expression for the total charge of the source. We see that

$$\int_{\Sigma^+} F^{ij}\,|_j\; d\Sigma_i = 4\pi \int_{\Sigma^+} J^i\, d\Sigma_i \quad . \tag{3.3.17}$$

We note that $\int_{r=\infty} F^{ij}\, d\Sigma_{ij} = 8\pi q$ and $(J^i)_{\Sigma^+} = 0$. So re-arranging (3.3.17) we have

$$Q_{TOTAL} = Q_{T^+} + Q_{DISC} \tag{3.3.18}$$

where the charge of on the disc and the exterior of the toroidal shell are given by

$$Q_{T^+} = \frac{1}{8\pi}\int_{T^+} F^{ij}\, d\Sigma_{ij} \tag{3.3.19}$$

$$Q_{DISC} = \frac{1}{8\pi}\int_{D^+} F^{ij}\, d\Sigma_{ij} + \frac{1}{8\pi}\int_{D^-} F^{ij}\, d\Sigma_{ij} \quad . \tag{3.3.20}$$

a) Contributions to mass of source

We find it convenient to introduce the following notation: we define $E = -N^2$, $I = -N_0^{n+2}/N^n$, and $\alpha = e^2/2a^2$. This makes it possible to express H as:

$$H_{EXTERIOR} = \cdots + \alpha\, E \qquad (\sigma < \sigma_0) \tag{3.3.21}$$

$$H_{INTERIOR} = \cdots + \alpha\, I \qquad (\sigma > \sigma_0) \tag{3.3.22}$$

where '$\cdots$' represents the value of H that we used in chapter two when examining the source of the Kerr metric.

The formulae for the mass of the disc and the annulus involve the quantities E_ψ and I_ψ, respectively [see equation (2.5.37)]; both these quantities are zero for

$\psi = \pm\pi$. Therefore, the mass of the disc and annulus are the same as for the Kerr source,

$$M_{DISC} = m - m\cosh\frac{\sigma_0}{2} \tag{3.3.23}$$

$$M_{ANNULUS} = -m\cosh\frac{\sigma_0}{2} \ . \tag{3.3.24}$$

The mass of the toroidal shell is given by the formula (2.5.47) which reduces to

$$M_{TORUS} = \frac{1}{2}\int_{-\pi}^{\pi} d\psi \,\{(\sqrt{-g}\,\xi^{t|\sigma})_{T^-} - (\sqrt{-g}\,\xi^{t|\sigma})_{T^+}\} \tag{3.3.25}$$

where equation (A.15) implies

$$(\sqrt{-g}\,\xi^{t|\sigma})_{T^+} = \cdots - 2a\alpha\frac{\sinh^2\frac{\sigma_0}{2}\cosh^2\frac{\sigma_0}{2}}{N_0^2} \tag{3.3.26}$$

$$(\sqrt{-g}\,\xi^{t|\sigma})_{T^-} = \cdots + na\alpha\frac{\sinh^2\frac{\sigma_0}{2}\cosh^2\frac{\sigma_0}{2}}{N_0^2} \ . \tag{3.3.27}$$

Therefore, we have

$$M_{TORUS} = \cdots - \alpha(1+\frac{n}{2})\sinh^2\frac{\sigma_0}{2}\cosh^2\frac{\sigma_0}{2}\int_{-\pi}^{\pi}\frac{d\psi}{N_0^2} \ . \tag{3.3.28}$$

Looking up our table of integrals, (C.1), we find

$$M_{TORUS} = 2m\cosh\frac{\sigma_0}{2} - \frac{\pi e^2}{2a}(1+\frac{n}{2})\sinh\frac{\sigma_0}{2}\cosh\frac{\sigma_0}{2} \ . \tag{3.3.29}$$

The mass due to the external electromagnetic field is

$$M_{EXTERIOR} = \frac{\alpha}{4\pi}\int_0^{2\pi}d\phi\int_{-\pi}^{\pi}d\psi\int_{\sigma_0}^{0}d\sigma\,\Box E \ . \tag{3.3.30}$$

This follows from (3.3.11) and (B.17). Using (B.10) to calculate $\Box E$, we get

$$\Box E = -\frac{2N^4}{a^2}(\cosh^2\frac{\sigma}{2} + \sinh^2\frac{\sigma}{2}) \ . \tag{3.3.31}$$

Integrating over the ψ and ϕ variables we obtain

$$M_{EXTERIOR} = \frac{\pi e^2}{2a} \int_0^{\sigma_0} d\sigma \, (\cosh^2 \frac{\sigma}{2} + \sinh^2 \frac{\sigma}{2}) \quad . \tag{3.3.32}$$

We make use of the hyperbolic identities, $\cosh 2\sigma = \cosh^2 \sigma + \sinh^2 \sigma$ and $\sinh 2\sigma = 2 \sinh \sigma \cosh \sigma$, to evaluate (3.3.32) as

$$M_{EXTERIOR} = \frac{\pi e^2}{2a} \cosh \frac{\sigma_0}{2} \sinh \frac{\sigma_0}{2} \quad . \tag{3.3.33}$$

The calculation of the mass of the interior region is a little bit trickier since $\Box I$ is not a simple expression for arbitrary n,

$$M_{INTERIOR} = \cdots + \frac{\alpha}{2} \int_{-\pi}^{\pi} d\psi \int_{\infty}^{\sigma_0} d\sigma \, \sqrt{-g} \, \Box I \quad . \tag{3.3.34}$$

We shall leave this calculation for now but will come back to it once we have looked at the angular momentum contributions.

b) Contributions to angular momentum of source

As might be expected the calculation of the angular momentum of the source is more involved than the preceding calculation for the mass. However, once more we need only concern ourselves with those contributions that involve α.

The angular momentum of the disc is given by equation (3.3.15). Equation (A.15) gives

$$\xi^{\psi|t} = \eta^{\psi\psi} \eta_{\phi\phi} v^{\phi}_{,\psi} H + v_{\phi} (\eta^{\psi\psi} H_{\psi}) \quad . \tag{3.3.35}$$

For $\psi = \pm\pi$ we have $v^{\phi}_{,\psi} = E_{\psi} = I_{\psi} = 0$. Therefore, the integrals over the disc and annulus do not contain any contributions containing α, and they give the same answers as in the source of the Kerr geometry;

$$J_{DISC} = -ma + \frac{ma}{2} \cosh \frac{\sigma_0}{2} + \frac{ma}{2} \frac{1}{\cosh \frac{\sigma_0}{2}} \tag{3.3.36}$$

$$J_{ANNULUS} = \frac{ma}{2}\cosh\frac{\sigma_0}{2} - \frac{ma}{6}\frac{1}{\cosh\frac{\sigma_0}{2}} \quad . \tag{3.3.37}$$

The angular momentum associated with the toroidal shell is

$$J_{TORUS} = -\frac{1}{4}\int_{-\pi}^{\pi} d\psi \{(\sqrt{-g}\xi^{t|\sigma})_{T^-}) - (\sqrt{-g}\xi^{t|\sigma})_{T^+})\} \quad . \tag{3.3.38}$$

Using the expressions for $(\xi^{t|\sigma})_{T^\pm}$ equation (3.3.38) reduces to

$$J_{TORUS} = -\frac{1}{4}\int_{-\pi}^{\pi} d\psi\,(\sqrt{-g}\,v_\phi\,\eta^{\sigma\sigma})_{\sigma=\sigma_0}\{(H_\sigma)_{T^-} - (H_\sigma)_{T^+}\} \quad . \tag{3.3.39}$$

Inserting all the relevant information into (3.3.39) we find

$$J_{TORUS} = \cdots + \frac{e^2}{8}(1+\frac{n}{2})\sinh^4\frac{\sigma_0}{2}\cosh^2\frac{\sigma_0}{2}\int_{-\pi}^{\pi}\frac{d\psi}{N_0^4} \quad . \tag{3.3.40}$$

Looking this integral up in our table of integrals, (C.2), the full expression for the angular momentum of the toroidal shell is

$$\begin{aligned} J_{TORUS} &= -ma\cosh\frac{\sigma_0}{2} + \frac{ma}{3}\frac{1}{\cosh\frac{\sigma_0}{2}} \\ &\quad + \frac{e^2\pi}{8}(1+\frac{n}{2})\frac{\sinh\frac{\sigma_0}{2}}{\cosh\frac{\sigma_0}{2}}(\cosh^2\frac{\sigma_0}{2} + \sinh^2\frac{\sigma_0}{2}) \quad . \end{aligned} \tag{3.3.41}$$

The exterior electromagnetic field also contributes to the angular momentum;

$$J_{EXTERIOR} = \frac{1}{8\pi}\int_0^{2\pi} d\phi \int_{-\pi}^{\pi} d\psi \int_{\sigma_0}^{0} d\sigma\,\sqrt{-g}\,R^t_\phi \quad . \tag{3.3.42}$$

By plugging (3.3.31), (B.11), and (B.3) into the equation for R^t_ϕ , (B.17), we find

$$-\sqrt{-g}\,R^t_\phi = \alpha a^2\,\frac{\sinh^3\frac{\sigma}{2}\cosh^3\frac{\sigma}{2}}{N^4} \quad . \tag{3.3.43}$$

Carrying out the integrals in (3.3.42) using the formula (C.2) we find that

$$J_{EXTERIOR} = -\frac{e^2\pi}{4}\sinh\frac{\sigma_0}{2}\cosh\frac{\sigma_0}{2} \quad . \tag{3.3.44}$$

The calculation of the angular momentum of the interior is quite involved. We start from the formula (3.3.14) which along with (B.16) gives

$$J_{INTERIOR} = -\frac{1}{8\pi}\int_0^{2\pi} d\phi \int_{-\pi}^{\pi} d\psi \int_{\infty}^{\sigma_0} d\sigma \sqrt{-g}\,(v_\phi \Box H + C_\phi) \quad . \qquad (3.3.45)$$

Before calculating the contributions that the interior region gives to both the mass and the angular momentum of the source, we should stop and apply some hindsight. We desire that our source have total mass, m, and total angular momentum, $-ma$. If this condition is to be true then we can deduce that the mass of the interior region must be

$$M_{INTERIOR} = -\frac{n\pi e^2}{4a}\sinh\frac{\sigma_0}{2}\cosh\frac{\sigma_0}{2} \qquad (3.3.46)$$

and that the angular momentum of the interior region must be

$$J_{INTERIOR} = \cdots + \frac{\pi e^2}{16}\frac{\sinh\frac{\sigma_0}{2}}{\cosh\frac{\sigma_0}{2}}\{2 - n(\cosh^2\frac{\sigma_0}{2} + \sinh^2\frac{\sigma_0}{2})\} \quad . \qquad (3.3.47)$$

We observe that the only way that $M_{INTERIOR}$ and $J_{INTERIOR}$ can be finite in the limit where σ_0 tends to infinity is if we choose n to be zero. We consider the above argument good enough to narrow down our choice of 'n' and from now on we take n to be zero. Therefore, the value of the scalar H in the interior region is

$$H = \cdots - \alpha N_0^2 \quad \text{for} \quad \sigma > \sigma_0 \quad . \qquad (3.3.48)$$

We should not rely on guess work alone. We must explicitly calculate the mass and angular momentum of the interior region.

The mass of the interior region is given by equation (3.3.34). For H given by (3.3.48) we find that

$$\sqrt{-g}\,\Box I = \frac{a}{2N^4}(\sinh^2\frac{\sigma}{2}\cos^2\frac{\psi}{2} - \cosh^2\frac{\sigma}{2}\sin^2\frac{\psi}{2})\sinh\frac{\sigma}{2}\cosh\frac{\sigma}{2} \quad . \qquad (3.3.49)$$

We carry out the integration over ψ using the formulae (C.6) and (C.7), and derive an answer of zero. Therefore, the mass of the interior region is exactly zero, that is

$$M_{INTERIOR} = 0 \quad . \tag{3.3.50}$$

The angular momentum of the interior region is given by equation (3.3.45). From equations (3.3.49) and (B.11) we find that

$$\begin{aligned} -\sqrt{-g}R^t_\phi \;=\; \cdots + \frac{a^2\alpha}{2N^6}\sinh^3\frac{\sigma}{2}\cosh\frac{\sigma}{2}\,[\cosh^2\frac{\sigma_0}{2} \\ + \cosh^2\frac{\sigma}{2}\,(\sin^2\frac{\psi}{2} - \cos^2\frac{\psi}{2})] \quad . \end{aligned} \tag{3.3.51}$$

Performing the integration over the ψ variable in equation (3.3.45) using the formulae (C.5), (C.6), (C.7), the equation for the angular momentum of the interior region becomes

$$\begin{aligned} J_{INTERIOR} \;=\; \cdots + \frac{a^2\alpha\pi}{32}\int_{\sigma_0}^{\infty} d\sigma\,[3\,\frac{\sinh^2\frac{\sigma_0}{2}}{\sinh^2\frac{\sigma}{2}} + \frac{2\sinh^2\frac{\sigma_0}{2}-1}{\cosh^2\frac{\sigma}{2}} \\ + 3\cosh^2\frac{\sigma_0}{2}\,\frac{\sinh^2\frac{\sigma}{2}}{\cosh^4\frac{\sigma}{2}}] \quad . \end{aligned} \tag{3.3.52}$$

Making the change of variable, $y = \tanh\frac{\sigma}{2}$, the integral becomes trivial

$$\begin{aligned} J_{INTERIOR} \;=\; \cdots + \frac{a^2\alpha\pi}{16}\int_{\tanh\frac{\sigma_0}{2}}^{1} dy\,[3\sinh^2\frac{\sigma_0}{2}\,(\frac{1}{y^2}) \\ + 2\sinh^2\frac{\sigma_0}{2} - 1 + 3\cosh^2\frac{\sigma_0}{2}\,(y^2)] \end{aligned} \tag{3.3.53}$$

Working out this integral and evaluating between the given limits we find that the angular momentum of the interior region is

$$J_{INTERIOR} = -\frac{2ma}{3}\frac{1}{\cosh\frac{\sigma_0}{2}} + \frac{e^2\pi}{8}\frac{\sinh\frac{\sigma_0}{2}}{\cosh\frac{\sigma_0}{2}} \quad . \tag{3.3.54}$$

As predicted the values of $M_{INTERIOR}$ and $J_{INTERIOR}$ are finite in the limit where σ_0 tends to infinity.

c) Electric charge of source

We express the electromagnetic potential for the Kerr-Newman metric as a differential 1-form and in Boyer-Lindquist [11] coordinates $(\bar{t}, \bar{\phi}, r, \theta)$ it is

$$A = -\frac{er}{\Sigma}\, d\bar{t} - \frac{aer}{\Sigma} \sin^2\theta\, d\bar{\phi} \quad . \tag{3.3.55}$$

We can write the electromagnetic potential in quasispheroidal coordinates [10] (t, ϕ, r, θ) as

$$A = -\frac{er}{\Sigma}\, dt - \frac{ear}{\Sigma} \sin^2\theta\, d\phi + \frac{2er^2 m_{\text{eff}}}{\Sigma\Delta}\left(1 - \frac{a^2}{r^2+a^2}\sin^2\theta\right) dr \tag{3.3.56}$$

where $\Delta = r^2 - 2m_{\text{eff}} r + a^2$. Quasispheroidal coordinates are related to Boyer-Lindquist coordinates by the following coordinate transformation;

$$d\phi = d\bar{\phi} - \frac{2am_{\text{eff}}}{r^2+a^2}\frac{1}{\Delta}\, dr \tag{3.3.57}$$

$$dt = d\bar{t} + \frac{2m_{\text{eff}} r}{\Delta}\, dr \quad . \tag{3.3.58}$$

The coordinates of choice are toroidal coordinates and here the electromagnetic potential has the not too pleasant form

$$A = -\frac{e}{a} N \cos\frac{\psi}{2}\, dt - e\frac{\sinh^2\frac{\sigma}{2}}{N}\cos\frac{\psi}{2}\, d\phi \tag{3.3.59}$$
$$- \frac{e\cos\frac{\psi}{2}}{2N^2\cosh\frac{\sigma}{2}} \frac{\frac{2ma\cos\frac{\psi}{2}}{N} - e^2}{\Delta}\left(\sinh\frac{\sigma}{2}\cos\frac{\psi}{2}\, d\sigma + \cosh\frac{\sigma}{2}\sin\frac{\psi}{2}\, d\psi\right)$$

where now $\Delta = \frac{a^2\cos^2\frac{\psi}{2}}{N^2} - \frac{2am\cos\frac{\psi}{2}}{N} + a^2 + e^2$.

We need to calculate F^{ij} and with the aid of the expression for the inverse of a Kerr-Schild metric (A.2) we have

$$F^{ij} = g^{ik} g^{jl} F_{kl} \tag{3.3.60}$$

$$F^{ij} = \eta^{ik}\eta^{jl} F_{kl} + 2H\,(v^k F_{kl})(\eta^{il} v^j - \eta^{jl} v^i) \quad . \tag{3.3.61}$$

If we perform the integration over ϕ then the charge of the disc is

$$Q_{DISC} = \frac{1}{2}\int_0^{\sigma_0} d\sigma\,\sqrt{-g}\,(F^{\psi t})_{D^+} + \frac{1}{2}\int_{\sigma_0}^{0} d\sigma\,\sqrt{-g}\,(F^{\psi t})_{D^-} \quad . \tag{3.3.62}$$

On D^+ the electric field 2-form F is

$$(F)_{D^+} = \frac{e}{2a}\cosh\frac{\sigma}{2}\,d\psi\wedge dt + \frac{e}{2}\frac{\sinh^2\frac{\sigma}{2}}{\cosh\frac{\sigma}{2}}\,d\psi\wedge d\phi \quad . \tag{3.3.63}$$

We also find that $(F)_{D^+} = -(F)_{D^-}$. Therefore, the non-zero components of the electromagnetic field tensor are

$$(F_{\psi t})_{D^\pm} = \pm\frac{e}{2a}\cosh\frac{\sigma}{2} \quad \text{and} \quad (F_{\psi\phi})_{D^\pm} = \pm e\frac{\sinh^2\frac{\sigma}{2}}{\cosh\frac{\sigma}{2}} \quad . \tag{3.3.64}$$

Using (3.3.61) and (3.3.64) we calculate $F^{\psi t}$ to be

$$(F^{\psi t})_{D^\pm} = \mp\frac{2e}{a^3}\cosh^5\frac{\sigma}{2} \quad . \tag{3.3.65}$$

We can now calculate the charge of the disc using equation (3.3.62) and we obtain

$$Q_{DISC} = e - e\cosh\frac{\sigma_0}{2} \quad . \tag{3.3.66}$$

Equation (3.3.19) tells us that the charge of the toroidal shell is

$$Q_{T^+} = \frac{1}{2}\int_{\pi}^{-\pi} d\psi\,\sqrt{-g}\,(F^{t\sigma})_{T^+} \quad . \tag{3.3.67}$$

Now equation (3.3.61) shows us how to calculate $F^{t\sigma}$,

$$F^{t\sigma} = \eta^{tt}\eta^{\sigma\sigma}F_{t\sigma} + 2H\left\{\eta^{tt}v^\sigma(v^i F_{it}) - \eta^{\sigma\sigma}v^t(v^i F_{i\sigma})\right\} \quad . \tag{3.3.68}$$

One of the first things that we notice about about the expression (3.3.68) is that the part involving H will contribute two terms, one proportional to m and the other proportional to e^3. With hindsight we expect that these terms will not

contribute to the charge of the toroidal shell. We can indeed show that to highest order in $e^{\sigma_0/2}$ we have

$$Q_{T+} = \frac{1}{2}\int_{\pi}^{-\pi} d\psi \sqrt{-g}\,\eta^{tt}\eta^{\sigma\sigma}\,(F_{t\sigma})_{T+} + \mathcal{O}(e^{-\sigma_0/2}) \quad . \tag{3.3.69}$$

Equations (3.3.60) and (3.3.16) tell us that

$$(F_{t\sigma})_{T+} = -\frac{e}{2a}\frac{\sinh\frac{\sigma_0}{2}\cosh\frac{\sigma_0}{2}}{N_0}\cos\frac{\psi}{2} \quad . \tag{3.3.70}$$

Therefore, we have

$$Q_{T+} = \frac{e}{4}\sinh^2\frac{\sigma_0}{2}\cosh^2\frac{\sigma_0}{2}\int_{-\pi}^{\pi} d\psi\frac{\cos\frac{\psi}{2}}{N_0^3} + \mathcal{O}(e^{-\sigma_0/2}) \quad . \tag{3.3.71}$$

Evaluating this integral [see equation (C.11)], the electric charge of the toroidal shell is

$$Q_{T+} = e\cosh\frac{\sigma_0}{2} + \mathcal{O}(e^{-\sigma_0/2}) \quad . \tag{3.3.72}$$

Adding Q_{DISC} and Q_{T+} together we find that our source has the correct charge,

$$Q_{TOTAL} = e + \mathcal{O}(e^{-\sigma_0/2}) \quad . \tag{3.3.73}$$

3.4 Summary

The source of the Kerr-Newman black hole geometry is very similar to that of the Kerr black hole. The source consists of a toroidal shell, a disc of matter external to the shell, and an annulus of matter interior to the toroidal shell. The torus is filled with some charged matter that gives zero contribution to the mass but a finite contribution to the angular momentum—the contribution to the angular momentum is of electromagnetic origin and is due to the presence of a non zero Poynting vector in the interior. The different contributions to the total mass and angular momentum are summarised in the following tables:

Mass contributions to the source	
M_{DISC}	$= m - m\cosh\frac{\sigma_0}{2}$
$M_{ANNULUS}$	$= -m\cosh\frac{\sigma_0}{2}$
M_{TORUS}	$= 2m\cosh\frac{\sigma_0}{2} - \frac{\pi\epsilon^2}{2a}\sinh\frac{\sigma_0}{2}\cosh\frac{\sigma_0}{2}$
$M_{EXTERIOR}$	$= \frac{\pi\epsilon^2}{2a}\sinh\frac{\sigma_0}{2}\cosh\frac{\sigma_0}{2}$
$M_{INTERIOR}$	$= 0$

These results are exact. We can see at a glance by adding all the mass contributions that we obtain the correct total mass of the source, namely m.

Angular momentum contributions to the source	
J_{DISC}	$= -ma + \frac{ma}{2}\cosh\frac{\sigma_0}{2} + \frac{ma}{2}\frac{1}{\cosh\frac{\sigma_0}{2}}$
$J_{ANNULUS}$	$= \frac{ma}{2}\cosh\frac{\sigma_0}{2} - \frac{ma}{6}\frac{1}{\cosh\frac{\sigma_0}{2}}$
J_{TORUS}	$= -ma\cosh\frac{\sigma_0}{2} + \frac{ma}{3}\frac{1}{\cosh\frac{\sigma_0}{2}} + \frac{\pi\epsilon^2}{8}\frac{\sinh\frac{\sigma_0}{2}}{\cosh\frac{\sigma_0}{2}}(\cosh^2\frac{\sigma_0}{2} + \sinh^2\frac{\sigma_0}{2})$
$J_{EXTERIOR}$	$= -\frac{\pi\epsilon^2}{4}\sinh\frac{\sigma_0}{2}\cosh\frac{\sigma_0}{2}$
$J_{INTERIOR}$	$= -\frac{2ma}{3}\frac{1}{\cosh\frac{\sigma_0}{2}} + \frac{\pi\epsilon^2}{8}\frac{\sinh\frac{\sigma_0}{2}}{\cosh\frac{\sigma_0}{2}}$

Again the total angular momentum of the source, $-ma$, is correct. To obtain the correct results we had to take into account the energy and angular momentum stored in the electromagnetic field. We also found that our source had

the correct charge.

We can show that the electric field lines exterior to the source are perpendicular to the disc but not to the toroidal shell. Therefore, the source is not a conductor. We do not know the exact form of the electromagnetic potential for the interior region, and it would be quite cumbersome to try and solve the Einstein-Maxwell field equations. However, it suffices to say that the non zero value of $J_{INTERIOR}$ in the limit that σ_0 is very large is probably due to the interior electromagnetic field. The other possibility is that the electromagnetic field in the interior is zero. This can be achieved by placing charges, dipoles, and other multipoles on the surface of the toroidal shell. We refrain from doing this since then the material stress-energy tensor of the interior region would have to contain terms proportional to e so as to give the correct angular momentum contribution.

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CHAPTER FOUR

GENERALISED SPLITTING OF SPACETIME

4.1 Introduction

The idea of spacetime splitting has been used extensively over the last few decades. Arnowitt, Deser and Misner (ADM)[1] were among the first people to realise the potential that this technique offered. Using the ADM 3+1 split they were able to disentangle the dynamical variables from the gauge variables in the general relativity field equations. They separated the metric field into two sets: one set consisting of those parts that carry true dynamical information, and the other set consisting of constraint equations that need to be satisfied on each hypersurface.

ADM considered the 4-geometry to consist of a foliation of 3-geometries. Spacetime is sliced into spacelike hypersurfaces with each hypersurface labeled by a global time parameter, so that successive hypersurfaces are labeled by successive values of the time parameter. A complete description of the global spacetime can be obtained by knowing the intrinsic 3-geometry of a particular hypersurface, and by knowing how that 3-geometry is embedded in the enveloping 4-geometry. The dynamical field equations tell how to find the 3-geometry on the other hypersurfaces thus giving a complete description of the 4-geometry.

Having separated out the time variable, the field equations can be cast into the Hamiltonian form. The Hamiltonian equations can be quantised using the canonical quantisation method to yield the Wheeler-De Witt equation. The ADM 3+1 split is a powerful tool for examining initial value and boundary value problems in general relativity, and for exploring a quantum theory of gravity.

This very useful technique of spacetime splitting can be extended beyond the usual 3+1 split. The foliating spaces in the ADM 3+1 split are hypersurfaces whose dimension is one less that that of the global spacetime which may not always be the most desirable choice and there are occasions when we wish to consider other spacetime splittings. For instance, a 2+2 splitting seems natural when dealing with cosmic strings; the two dimensional world history of the cosmic string can be selected out. An $m + n$ splitting of spacetime is often helpful for gravitational theories that can be cast in a Kaluza Klein form.

In this chapter, we consider a foliation where the dimension of the foliating spaces can be any natural number less than the dimension of the global space. We can still discuss the intrinsic geometry of the spaces in much the same way as in the ADM formalism but with a few added features. Only one normal vector exists in the ADM formalism at any point on a given hypersurface. We can observe how this hypersurface is embedded in the global spacetime by noting how the normal varies over any given hypersurface. This embedding information is contained in a quantity called the extrinsic curvature tensor. The possibility of more than one independent normal vector exists for our generalised foliation and this in turn leads to more than one extrinsic curvature tensor; we refer to them as generalised extrinsic curvature tensors. Given a set of foliating spaces we still need to prescribe how to carry our local intrinsic coordinates from one space to another; this information is contained in the shift vectors. Only one shift vector is present in the ADM formalism but there may be several in the generalised spacetime splitting formalism. The ADM lapse function, which is related to the proper distance between two neighbouring hypersurfaces, must be generalised to a lapse matrix. The field equations can be expressed in terms of tangential and normal components to the foliating spaces. These equations are the extension of

the usual ADM dynamical and constraint equations.

In section two, we give a review of the ADM 3+1 formalism. The definitions of the lapse function, the shift vector, and the extrinsic curvature tensor are given, and we show how to derive the Gauss-Weingarten and the Gauss-Codazzi equations, along with the projected components of the Ricci tensor. The derivation is detailed so that the reader can get a clear picture of how the generalised spacetime splitting is related to the ADM splitting.

Section three, contains a formal description of what we mean by an $m+n$ splitting of spacetime. We define analogous quantities to those defined in section two, and introduce some new quantities that do not arise in the ADM 3+1 split. The full derivation for the generalised Gauss-Weingarten, Gauss-Codazzi and projected components of the Ricci tensor are given.

In section four, we show how the generalised formalism reduces to the ADM formalism, when the 3+1 split is examined. This is not as obvious as it looks since our normal vectors are not necessarily unit vectors unlike in the ADM case.

Finally, in section five, a special case of the generalised splitting is examined. We look at an orthogonal split and find that the generalised equations reduce to an elegant symmetric form.

4.2 ADM 3+1 Split

One of the initial motivations for formulating a 3+1 split of space-time is the analysis of the initial value problem. Given a complete set of Cauchy data we can uniquely solve the field equations. We need to be able to extract the dynamical equations which tell us how our solution evolves and the constraint equations

which must be imposed to satisfy the initial conditions.

We consider a pseudo-Riemannian manifold, $\mathcal{M}$, which is foliated by a family of hypersurfaces. Individual hypersurfaces are labeled by the parameter t and are denote by Σ_t. Each hypersurface is a submanifold. Due to the foliation we can write $\mathcal{M} = \{\Sigma_t : t \epsilon R\}$. Picking one Σ_t in particular, we can describe it by the parametric equation, $X^\mu = X^\mu(\xi^a)$. The $\{\xi^a\}$ are intrinsic coordinates for Σ_t, whereas $\{X^\mu\}$ are extrinsic coordinates for the manifold $\mathcal{M}$. Comparing two points on Σ_t that are infinitesimally close to each other, we have

$$dX^\mu = e^\mu_{(a)}\, d\xi^a \quad \text{where} \quad e^\mu_{(a)} = \frac{\partial X^\mu}{\partial \xi^a} \quad . \tag{4.2.1}$$

The $e^\mu_{(a)}$ are called base vectors and they lie in Σ_t. The line element on Σ_t is

$$ds^2 = g_{\mu\nu}\, dX^\mu dX^\nu = g_{ab}\, d\xi^a d\xi^b \tag{4.2.2}$$

where we have defined $g_{ab} = g_{\mu\nu}\, e^\mu_{(a)}\, e^\nu_{(b)} = e_{(a)} \cdot e_{(b)}$. Therefore, g_{ab} are the components of the intrinsic metric on Σ_t. We assume that the hypersurfaces are either timelike or spacelike in the ADM 3+1 split. This guarantees that the determinant of g_{ab} is not equal to zero on any hypersurface Σ_t and we can define its inverse, g^{ab}, such that $g^{ab} g_{bc} = \delta^a_c$. Let n^α be the unit normal to Σ_t,

$$\text{that is} \quad n^\alpha n_\alpha = \epsilon \quad \text{where} \quad \epsilon = \begin{cases} +1 & \text{for a spacelike } n^\alpha \\ -1 & \text{for a timelike } n^\alpha \end{cases} \quad . \tag{4.2.3}$$

We define

$$N^\alpha = \frac{\partial X^\alpha}{\partial t} \tag{4.2.4}$$

and we find that in general the differential dX^α is

$$dX^\alpha = e^\alpha_{(a)}\, d\xi^a + N^\alpha\, dt \quad . \tag{4.2.5}$$

The vector N^α can be written in terms of n^α and $e^\alpha_{(a)}$ as follows:

$$N^\alpha = N n^\alpha + N^a e^\alpha_{(a)} \quad . \tag{4.2.6}$$

We call N the lapse function and N^a the shift vector. If we have two nearby hypersurfaces Σ_t and $\Sigma_{t+\delta t}$, then $N^\mu \delta t$ is the vector connecting points that have the same intrinsic coordinates and $N\delta t$ is the proper orthogonal distance between the hypersurfaces. The shift vector tells us how far the intrinsic coordinates have been moved along the surface from the normal direction, see figure 4.1.

The full line element is

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dX^\mu dX^\nu = g_{\mu\nu}\,(e^\mu_{(a)}\, d\xi^a + N^\mu\, dt)\,(e^\nu_{(b)}\, d\xi^b + N^\nu\, dt) \\ ds^2 &= (\epsilon N^2 + g_{ab}\, N^a\, N^b)\, dt^2 + 2\, g_{ab}\, N^b\, d\xi^a dt + g_{ab}\, d\xi^a d\xi^b \quad . \end{aligned} \tag{4.2.7}$$

For convenience, the intrinsic metric is employed to raise and lower the indices of those quantities defined on each hypersurface. For instance we use N_a to denote $g_{ab}\, N^b$ and likewise $N^b = g^{ab}\, N_a$. With this notation in mind

$$ds^2 = (\epsilon N^2 + N_a N^a)\, dt^2 + 2\, N_a\, d\xi^a dt + g_{ab}\, d\xi^a d\xi^b \quad . \tag{4.2.8}$$

We can also write the completeness relation for the metric in terms of its component

$$g^{\mu\nu} = g^{ab}\, e^\mu_{(a)}\, e^\nu_{(b)} + \epsilon\, n^\mu\, n^\nu \quad . \tag{4.2.9}$$

We define absolute derivatives $\frac{\delta}{\delta\xi^a}$ and $\frac{\delta}{\delta t}$ via

$$\begin{aligned} \frac{\delta}{\delta\xi^a}(\ldots) &= \frac{\partial X^\alpha}{\partial \xi^a}\frac{\delta}{\delta X^\alpha}(\ldots) = (\ldots)|_\alpha\, e^\alpha_{(a)} \\ \frac{\delta}{\delta t}(\ldots) &= \frac{\partial X^\alpha}{\partial t}\frac{\delta}{\delta X^\alpha}(\ldots) = (\ldots)|_\alpha\, N^\alpha \quad . \end{aligned} \tag{4.2.10}$$

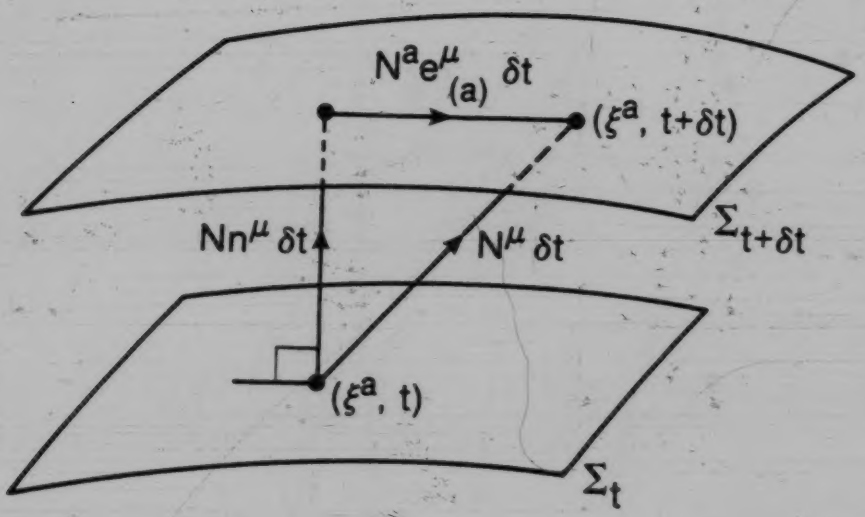

Figure 4.1: A spacetime diagram illustrating two neighbouring hypersurfaces Σ_t and $\Sigma_{t+\delta t}$. The geometric interpretation of the lapse function and the shift vector is indicated.

We are interested in how the base vectors and the normal vector vary as we move along a given surface or as we move from one surface to another. The variations $\frac{\delta n^\alpha}{\delta \xi^a}$ are parallel to Σ which can easily be seen from the fact that

$$2\, n_\alpha \frac{\delta n^\alpha}{\delta \xi^a} = \frac{\partial}{\partial \xi^a}\{n.n\} = 0\,. \tag{4.2.11}$$

Expressing $\frac{\delta n^\alpha}{\delta \xi^a}$ in terms of the base vectors, we write

$$\frac{\delta n^\alpha}{\delta \xi^a} = K_a{}^b\, e^\alpha_{(b)} \tag{4.2.12}$$

which defines the extrinsic curvature 3-tensor K_{ab} for Σ. The extrinsic curvature is a measure of the curvature ("bending") of Σ relative to the enveloping manifold $\mathcal{M}$. Explicitly we have

$$K_{ab} = n_{\alpha|\beta}\, e^\beta_{(a)}\, e^\alpha_{(b)} = -\, e^\alpha_{(b)|\beta}\; e^\beta_{(a)}\, n_\alpha = -\, e^\alpha_{(a)|\beta}\; e^\beta_{(b)}\, n_\alpha = K_{ba}\quad , \tag{4.2.13}$$

that is, the extrinsic curvature tensor is symmetric. Our convention for the extrinsic curvature is the same as Israel[2] but opposite to ADM[1], De Witt[3], and Misner, Thorne, and Wheeler[4]. The other surface deviation equation is

$$\frac{\delta e^\alpha_{(a)}}{\delta \xi^b} = -\,\epsilon\, K_{ab}\, n^\alpha + \Gamma^c_{ab}\, e^\alpha_{(c)}\quad . \tag{4.2.14}$$

Equation (4.2.14) defines Γ^c_{ab}. As the notation suggests Γ^c_{ab} is actually the Christoffel symbol associated with the intrinsic 3-metric g_{ab} which can be seen by noting

$$\begin{aligned} g_{ab,c} &= \frac{\delta}{\delta \xi^c}\{e_{(a)}.e_{(b)}\} = e_{(a)}.\frac{\delta e_{(b)}}{\delta \xi^c} + e_{(b)}.\frac{\delta e_{(a)}}{\delta \xi^c} \\ g_{ab,c} &= g_{ad}\,\Gamma^d_{bc} + g_{bd}\,\Gamma^d_{ac}\quad . \end{aligned} \tag{4.2.15}$$

Following the usual procedure of writing down the cyclic permutations of equation (4.2.15) and observing that Γ^a_{bc} is symmetric in its lower two indices we find

$$\Gamma^a_{bc} = \frac{1}{2}\, g^{ad}\,(g_{bd,c} + g_{cd,b} - g_{bc,d})\,. \tag{4.2.16}$$

Therefore, Γ^a_{bc} is indeed a Christoffel symbol. We define covariant and contravariant derivatives on Σ by

$$v_{a;b} = v_{a,b} - \Gamma^c_{ab} v_c \tag{4.2.17}$$

$$v^a_{;b} = g^{ac} v_{c;b} = v^a_{,b} + v^c \Gamma^a_{bc} \ . \tag{4.2.18}$$

Turning now to variations that take us from one hypersurface to another, we have

$$\frac{\delta e^a_{(a)}}{\delta t} \equiv \frac{\delta N^a}{\delta \xi^a} = A_a n^a + A_a{}^b e^a_{(b)} \ . \tag{4.2.19}$$

Using the definition (4.2.6) in combination with the deviation equations (4.2.12) and (4.2.14), we can derive expressions for A_a and $A_a{}^b$

$$A_a = N_{,a} - \epsilon K_{ab} N^b \tag{4.2.20}$$

$$A_a{}^b = N^b_{;a} + N K_a{}^b \ . \tag{4.2.21}$$

The final deviation equation is

$$\frac{\delta n^a}{\delta t} = -\epsilon A^a e^a_{(a)} \ . \tag{4.2.22}$$

We made use of the fact that $A_a = \epsilon n . \frac{\delta e_{(a)}}{\delta t} = -\epsilon e_{(a)} . \frac{\delta n}{\delta t}$ to derive (4.2.22). The equations (4.2.12), (4.2.14), (4.2.19), and (4.2.22) are collectively known as the Gauss-Weingarten equations.

We now turn our attention to the integrability conditions for the Gauss-Weingarten equations. These are essentially the Ricci commutation relations which read

$$(\frac{\delta}{\delta \xi^c}\frac{\delta}{\delta \xi^b} - \frac{\delta}{\delta \xi^b}\frac{\delta}{\delta \xi^c}) v^a = -R^a_{\lambda\beta\gamma} v^\lambda e^\beta_{(b)} e^\gamma_{(c)} \tag{4.2.23}$$

$$(\frac{\delta}{\delta t}\frac{\delta}{\delta \xi^a} - \frac{\delta}{\delta \xi^a}\frac{\delta}{\delta t}) v^a = -R^a_{\lambda\beta\gamma} v^\lambda e^\beta_{(a)} N^\gamma \ . \tag{4.2.24}$$

Taking $v^\alpha = e^\alpha_{(a)}$ in (4.2.23) and making repeated use of (4.2.12) and (4.2.14), we can explicitly evaluate the left hand side of equation (4.2.23) to obtain

$$R_{\alpha\beta\gamma\lambda}\, e^\alpha_{(d)}\, e^\beta_{(a)}\, e^\gamma_{(b)}\, e^\lambda_{(c)} = R_{dabc} + 2\,\epsilon\, K_{a[b}\, K_{c]d} \tag{4.2.25}$$

$$R_{\alpha\beta\gamma\lambda}\, n^\alpha\, e^\beta_{(a)}\, e^\gamma_{(b)}\, e^\lambda_{(c)} = 2\, K_{a[b;c]} \ . \tag{4.2.26}$$

With $v^\alpha = e^\alpha_{(a)}$ and using all the Gauss-Weingarten equations, we can evaluate the left hand side of equation (4.2.24). The right hand side of (4.2.24) can be expanded using (4.2.6) and (4.2.26). Comparing the two results we are left with

$$N\, R_{\alpha\beta\gamma\lambda}\, n^\alpha\, e^\beta_{(a)}\, e^\gamma_{(b)}\, n^\lambda = -2\, N^c\, K_{a[b;c]} + \dot{K}_{ab} - K_b{}^c A_{ac} + \epsilon\, A_{a;b} \ . \tag{4.2.27}$$

Inserting the expressions (4.2.20) and (4.2.21) into (4.2.27) we find

$$N\, R_{\alpha\beta\gamma\lambda}\, n^\alpha\, e^\beta_{(a)}\, e^\gamma_{(b)}\, n^\lambda = \dot{K}_{ab} - K_{ab;c}\, N^c - 2\, N^c_{;(a} K_{b)c} + \epsilon\, N_{;ab} - N\, K_{ac}\, K_b{}^c \ . \tag{4.2.28}$$

Equations (4.2.25), (4.2.26), and (4.2.28) are commonly known as the Gauss-Codazzi equations. Finally, using the completeness relation for the metric, (4.2.9), we obtain the projected components of the Ricci tensor. Before proceeding any further we shall introduce the following notation, which is similar to specifying the frame components, that is, $\hat{R}_{a\ldots} = e^\alpha_{(a)}\, R_{\alpha\ldots}$ and $\hat{R}_{n\ldots} = n^\alpha\, R_{\alpha\ldots}$. We observe that

$$\begin{aligned}
\hat{R}_{ab} &= R_{\alpha\beta}\, e^\alpha_{(a)}\, e^\beta_{(b)} = g^{\gamma\delta}\, R_{\gamma\alpha\delta\beta}\, e^\alpha_{(a)}\, e^\beta_{(b)} \\
&= g^{cd}\, \hat{R}_{cadb} + \epsilon\, \hat{R}_{nanb} \\
&= R_{ab} + 2\,\epsilon\, K_a{}^c\, K_{bc} - \epsilon\, K\, K_{ab} \\
&\quad + \epsilon\, N^{-1} \{ K_{ab;c}\, N^c - \dot{K}_{ab} + 2\, N_{c;(b}\, K^c_{a)} - \epsilon\, N_{;ab} \} \ .
\end{aligned} \tag{4.2.29}$$

Here, $R_{ab} = g^{cd}\, R_{cadb}$ is the Ricci tensor for the intrinsic 3-metric g_{ab} and $K = g^{ab}\, K_{ab}$ is the trace of the extrinsic curvature of Σ. The remaining components of

the projected Ricci tensor are

$$\hat{R}_{an} = g^{bc}\,\hat{R}_{bacn} = K^{b}{}_{a;b} - K_{,a} \tag{4.2.30}$$

and

$$\hat{R}_{nn} = g^{ab}\,\hat{R}_{anbn} = -K_{ab}\,K^{ab} + N^{-1}\{K_{,a}\,N^{a} - \dot{K} - \epsilon N_{;a}{}^{a}\} \quad . \tag{4.2.31}$$

We used $\dot{g}^{ab} = -g^{ac}\,g^{bd}\,\dot{g}_{cd}$ in conjunction with

$$\dot{g}_{ab} = \frac{\partial}{\partial t}\{e_{(a)}\cdot e_{(b)}\} = A_{ab} + A_{ba} = 2\,N_{(a;b)} + 2\,N\,K_{ab} \tag{4.2.32}$$

to derive equation (4.2.31). Another quantity of interest to us is the Ricci scalar. We can easily compute this since

$$R = g^{\alpha\beta}\,R_{\alpha\beta} = g^{ab}\,\hat{R}_{ab} + \epsilon\,\hat{R}_{nn} \tag{4.2.33}$$

$$= {}^{3}R - \epsilon\,K^{2} - \epsilon\,K_{ab}\,K^{ab} + 2\epsilon\,N^{-1}\{-\dot{K} - \epsilon N_{;a}{}^{a} + K_{,a}\,N^{a}\} \quad . \tag{4.2.34}$$

That finishes our review of the ADM 3+1 split and we now move on to the generalised Gauss-Codazzi equations.

4.3 Generalised Gauss-Codazzi equations

We now consider the more general case of an $m+n$ dimensional pseudo-Riemannian manifold, $\mathcal{M}$. Points have coordinate values X^{μ} in a local coordinate system and the metric tensor has components $g_{\mu\nu}$ where μ, ν can take on values $1, 2, ..., m+n$ (m and n are positive integers). The usual approach is to consider a foliation of the spacetime into hypersurfaces of dimension one less than that of the manifold. We shall relax this condition and examine a foliation where the dimension of the

foliating spaces is arbitrary, m say. We call these spaces "*horizontal spaces.*" Our aim is to write down the embedding relationships so as to produce a working formulation. We are led to introduce intrinsic coordinates, $\{\xi^a\}$, on each horizontal space, $a = 1...m$. This is not the only approach to the problem. Carter[5] gives a beautiful mathematical formulation where only global tensors and coordinates are employed. To distinguish the different horizontal spaces, we label each horizontal space by a set of parameters $\{\zeta^A\}$ where $A = 1...n$, and we denote them by $\mathcal{H}(\zeta^A)$. The parameters $\{\zeta^A\}$ evolve continuously as one moves from one horizontal space to another. The manifold $\mathcal{M}$ consists of all the possible horizontal spaces, that is $\mathcal{M} = \{\mathcal{H}(\zeta^A) : \zeta^A \epsilon R^n\}$. The spaces given by $\xi^a = constant$ we call "*slanted spaces*" and they have intrinsic coordinates ζ^A.

We define the following sets of vectors:

$$\begin{aligned} 1)\quad e^{\mu}_{(a)} &= \frac{\partial X^\mu}{\partial \xi^a}\ , \quad e^{\mu}_{(A)} = \frac{\partial X^\mu}{\partial \zeta^A} \\ 2)\quad e^{(A)}_{\mu} &= \frac{\partial \zeta^A}{\partial X^\mu}\ , \quad e^{(a)}_{\mu} = \frac{\partial \xi^a}{\partial X^\mu}\ . \end{aligned} \tag{4.3.35}$$

Any vector v^μ can be written as a linear combination of $e^{\mu}_{(a)}$ and $e^{\mu}_{(A)}$. So, the former set form a basis for vector fields on $\mathcal{M}$. To preserve the resemblance to the ADM 3+1 split it is convenient to work with what we call the *fundamental basis* (or the "mixed basis") $\{e^{\mu}_{(a)}, e^{(A)\mu}\}$. The $e^{\mu}_{(a)}$ we again call base vectors and they span the horizontal spaces. The $\{e^{(A)\mu}\}$ form a basis for a space which we call the "*vertical space.*" The choice of nomenclature is prompted by the fact that the vertical space is orthogonal to the horizontal space in the sense that the respective basis vectors are orthogonal, that is, $e^{\mu}_{(a)}\, e^{(A)}_{\mu} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial \zeta^A}{\partial X^\mu} = \frac{\partial \zeta^A}{\partial \xi^a} = 0$. The $\{e^{(A)\mu}\}$ play the role of normal vectors but we no longer restrict ourselves to one independent normal vector.

We define $\gamma_{ab} = e_{(a)}.e_{(b)}$ and $\gamma^{AB} = e^{(A)}.e^{(B)}$. Therefore, the γ_{ab}

are the components of the intrinsic metric on $\mathcal{H}(\zeta^A)$. The hypersurfaces Σ_t in the ADM 3+1 split are chosen to be timelike or spacelike, in other words Σ_t is a non null hypersurface. We can carry this idea over to our $m+n$ split by modifying it slightly. We assume that the intrinsic metric of $\mathcal{H}(\zeta^A)$ is non-degenerate, that is, it has a non-vanishing determinant at every point of $\mathcal{M}$. This last assumption enables us to define the inverse intrinsic metric, γ^{ab}, in the usual manner. The γ^{AB} play an analogous role to ϵ in the ADM 3+1 split (where ϵ is ± 1). The assumption that the $\det(\gamma_{ab} \neq 0$ also implies that the determinant of γ^{AB} is non-zero everywhere. This can be seen by examining the completeness relationship for the metric [see equation (4.3.40)]. We write γ_{AB} for its inverse.

The reciprocal fundamental basis $\{e^{(a)\mu}\,,\, e^{\mu}_{(A)}\}$ is related to the fundamental basis by

$$e^{(a)\mu} = \gamma^{ab}\, e^{\mu}_{(b)} - \gamma^{a}_{B}\, e^{(B)\mu} \tag{4.3.36}$$

$$e^{\mu}_{(A)} = \gamma^{a}_{A}\, e^{\mu}_{(a)} + \gamma_{AB}\, e^{(B)\mu} \tag{4.3.37}$$

where

$$\gamma^{a}_{B} = -\gamma_{AB}\, e^{(A)}.e^{(a)} \tag{4.3.38}$$

$$= \gamma^{ab}\, e_{(b)}.e_{(B)} \quad . \tag{4.3.39}$$

We can see that (4.3.38) and (4.3.39) are equivalent since contracting equation (4.3.36) with $e_{(B)\mu}$ and $e^{(D)}_{\mu}$ gives (4.3.39) and (4.3.38), respectively. Comparing equation (4.3.37) to (4.2.6), we can think of γ_{AB} as a lapse matrix (we must be careful since we did not unitise $e^{(A)}_{\mu}$), and the γ^{a}_{A} as a set of n shift vectors.

The completeness relationship for the metric in terms of the fundamental basis has the simple form

$$g^{\mu\nu} = \gamma^{ab}\, e^{\mu}_{(a)} e^{\nu}_{(b)} + \gamma_{AB}\, e^{(A)\mu} e^{(B)\nu} \tag{4.3.40}$$

and the line element is

$$ds^2 = (\gamma_{AB} + \gamma_{ab}\,\gamma^a_A\,\gamma^b_B)\,d\zeta^A d\zeta^B + 2\,\gamma_{ab}\,\gamma^b_A\,d\zeta^A d\xi^a + \gamma_{ab}\,d\xi^a d\xi^b \quad . \qquad (4.3.41)$$

Our goal is to find the generalised Gauss-Weingarten and Gauss-Codazzi equations. We shall start by writing down the deviation equations that hold on the horizontal surfaces. These equations have the following form:

$$\frac{\delta e^{\alpha}_{(a)}}{\delta \xi^b} = -K_{Aab}\,e^{(A)\alpha} + \Gamma^c_{ab}\,e^{\alpha}_{(c)} \qquad (4.3.42)$$

$$\frac{\delta e^{(A)\alpha}}{\delta \xi^b} = K^{A}{}_{b}{}^{c}\,e^{\alpha}_{(c)} - P_{bB}{}^{A}\,e^{(B)\alpha} \quad . \qquad (4.3.43)$$

The n quantities K_{Aab} are similar to the extrinsic curvatures and are defined as

$$K_{Aab} = \gamma_{AB}\,e^{(B)}_{\alpha|\beta}\,e^{\beta}_{(b)}\,e^{\alpha}_{(a)} \quad . \qquad (4.3.44)$$

We can see that K_{Aab} is symmetric in its lower case roman indices. The definition of Γ^c_{ab} is unchanged and it remains the Christoffel symbol associated with the intrinsic metric of the horizontal space. Now we use γ_{ab} and γ^{ab} to lower and raise lower case roman indices. We introduce a similar practice for γ_{AB} and γ^{AB} , but with respect to upper case roman indices. As a result, we have

$$K^{A}{}_{b}{}^{c} \equiv \gamma^{AB}\,\gamma^{cd}\,K_{Bbd} \quad . \qquad (4.3.45)$$

The $P_{bB}{}^{A}$ term arises essentially due to the fact that the $e^{(A)\alpha}$ are not unit orthogonal vectors, that is, $e^{(A)}.e^{(B)} \neq \delta^{AB}$. We have

$$P_{bB}{}^{A} = -\gamma_{BD}\,e^{(A)}_{\alpha|\beta}\,e^{\beta}_{(b)}\,e^{(D)\alpha} \quad . \qquad (4.3.46)$$

This satisfies some useful identities which we will need later on, namely:

$$\gamma^{AB}{}_{,a} = -2\,P_a{}^{(AB)} \qquad (4.3.47)$$

$$\gamma_{AB,a} = 2\,P_{a(AB)} \quad . \qquad (4.3.48)$$

The identity (4.3.47) follows directly from (4.3.43) since $\gamma^{AB}{}_{,a} = \frac{\delta}{\delta\xi^a}\{e^{(A)}.e^{(B)}\}$.

For the deviations with respect to ζ^A we write

$$\frac{\delta e^{\alpha}_{(a)}}{\delta\zeta^A} = I\!K_A{}^b{}_a\, e^{\alpha}_{(b)} + I\!P_{aAB}\, e^{(B)\alpha} \tag{4.3.49}$$

$$\frac{\delta e^{(B)\alpha}}{\delta\zeta^A} = -I\!P^a{}_A{}^B\, e^{\alpha}_{(a)} - I\!\Gamma^B_{AD}\, e^{(D)\alpha} \ . \tag{4.3.50}$$

The equations (4.3.42), (4.3.43), (4.3.49) and (4.3.50) we call the generalised Gauss-Weingarten equations. The $I\!K_{Aba}$ and $I\!P_{aAB}$ are related to K_{Aab} , P_{aAB}, and γ^a_A . We note that

$$I\!K_{Aba} = e_{(b)}.\frac{\delta e_{(a)}}{\delta\zeta^A} = e_{(b)}.\frac{\delta e_{(A)}}{\delta\xi^a} \ . \tag{4.3.51}$$

Replacing $e^{\alpha}_{(A)}$ by the expression (4.3.37) and carrying out the deviation we find

$$I\!K_{Aab} = K_{Aab} + \gamma_{Aa;b} \ . \tag{4.3.52}$$

The following two identities are satisfied by $I\!K_{Aab}$:

$$\gamma_{ab,A} = 2\, I\!K_{A(ab)} \text{ and } \gamma^{ab}{}_{,A} = -2\, I\!K_A{}^{(ab)} \ . \tag{4.3.53}$$

Likewise, we have

$$I\!P_{aA}{}^D = e^{(D)}.\frac{\delta e_{(a)}}{\delta\zeta^A} = e^{(D)}.\frac{\delta e_{(A)}}{\delta\xi^a} \tag{4.3.54}$$

and using the same substitution as before we get

$$I\!P_{aAD} = P_{aAD} + \gamma^b_A\, K_{Dba} \ . \tag{4.3.55}$$

And finally, we define $I\!\Gamma^B_{AD}$ by

$$I\!\Gamma^B_{AD} = -\gamma_{DE}\, e^{(B)}_{\alpha|\beta}\, e^{\beta}_{(A)}\, e^{(E)\alpha} \ . \tag{4.3.56}$$

This $I\!\Gamma^B_{AD}$, contrary to our suggestive notation, is not the Christoffel symbol for γ_{AB}. It is not even symmetric in its lower two indices. However, we are still able

to define a covariant and a contravariant derivative if we are careful about the ordering of the indices. We can obtain some idea on how to proceed by examing the following identity:

$$\gamma^{AB}{}_{,D} = \frac{\delta}{\delta\zeta^D}\{e^{(A)}.e^{(B)}\} = -\gamma^{AE}\,\mathbb{\Gamma}^B_{DE} - \gamma^{BE}\,\mathbb{\Gamma}^A_{DE} \quad . \tag{4.3.57}$$

Re-arranging the equation in a more transparent form we get

$$\gamma^{AB}{}_{,D} + \gamma^{AE}\,\mathbb{\Gamma}^B_{DE} + \gamma^{BE}\,\mathbb{\Gamma}^A_{DE} = 0 \quad . \tag{4.3.58}$$

In this form the equation has shades of $\gamma^{ab}{}_{;c} = 0$, and we are prompted to define

$$\mathbb{L}^A{}_{\|B} = \mathbb{L}^A{}_{,B} + \mathbb{\Gamma}^A_{BD}\,\mathbb{L}^D \quad , \tag{4.3.59}$$

thus ensuring that $\gamma^{AB}\|_D = 0$. Multiplying (4.3.58) by $\gamma_{AF}\,\gamma_{BG}$ gives us $\gamma_{FG\|D} = 0$, where we have defined

$$\mathbb{L}_{A\|B} = \mathbb{L}_{A,B} - \mathbb{\Gamma}^D_{BA}\,\mathbb{L}_D \quad . \tag{4.3.60}$$

Note that we have preserved the fact that $\mathbb{L}_{A\|B} = \gamma_{AD}\,\mathbb{L}^D{}_{\|B}$. Also the above definition of differentiation satisfies the identity $(\mathbb{L}^A\mathbb{K}_A)_{\|B} = (\mathbb{L}^A\mathbb{K}_A)_{,B}$. Under a reparametrization $\zeta^A \to \zeta^{B'} = \zeta^{B'}(\zeta^A)$, (4.3.59) and (4.3.60) transform as tensors. So we are justified in using the terminology contravariant and covariant derivatives.

The generalised Gauss-Codazzi equations can be derived by proceeding along the same lines as in the preceding section. By expanding the Ricci commutation relations

$$(\frac{\delta}{\delta\xi^c}\frac{\delta}{\delta\xi^b} - \frac{\delta}{\delta\xi^b}\frac{\delta}{\delta\xi^c})\,e^\alpha_{(a)} = -R^\alpha_{\lambda\beta\gamma}\,e^\lambda_{(a)}\,e^\beta_{(b)}\,e^\gamma_{(c)} \tag{4.3.61}$$

$$(\frac{\delta}{\delta\zeta^A}\frac{\delta}{\delta\xi^a} - \frac{\delta}{\delta\xi^a}\frac{\delta}{\delta\zeta^A})\,e^{(B)\alpha} = -R^\alpha_{\lambda\beta\gamma}\,e^{(B)\lambda}\,e^\beta_{(a)}\,e^\gamma_{(A)} \tag{4.3.62}$$

$$(\frac{\delta}{\delta\zeta^A}\frac{\delta}{\delta\zeta^B} - \frac{\delta}{\delta\zeta^B}\frac{\delta}{\delta\zeta^A})\,e^{(D)\alpha} = -R^\alpha_{\lambda\beta\gamma}\,e^{(D)\lambda}\,e^\beta_{(B)}\,e^\gamma_{(A)} \tag{4.3.63}$$

we find after a not too long calculation

$$\hat{R}_{dacb} = R_{dacb} + 2\,K_{Aa[c}\,K^{A}{}_{b]d} \tag{4.3.64}$$

$$\hat{R}^{D}{}_{abc} = 2\,K^{D}{}_{a[b;c]} + 2\,K_{Aa[b}\,P_{c]}{}^{AD} \tag{4.3.65}$$

$$\hat{R}^{B}{}_{baA} = K^{B}{}_{ab\|A} - I\!K_{Acb}\,K^{B}{}_{a}{}^{c} + P_{aD}{}^{B}\,I\!P_{bA}{}^{D} + I\!P_{bA}{}^{B}{}_{;a} \tag{4.3.66}$$

$$\hat{R}^{BD}{}_{aA} = \gamma^{DE}\,I\!\Gamma^{B}_{AE,a} - P_{a}{}^{DB}{}_{\|A} + K^{B}{}_{a}{}^{b}\,I\!P_{bA}{}^{D} - K^{D}{}_{a}{}^{b}\,I\!P_{bA}{}^{B} \tag{4.3.67}$$

$$\hat{R}^{D}{}_{aBA} = -2\,I\!P_{a[B}{}^{D}{}_{\|A]} + 2\,I\!P_{aE}{}^{D}\,I\!\Gamma^{E}_{[BA]} + 2\,I\!P^{b}{}_{[B}{}^{D}\,I\!K_{A]ba} \tag{4.3.68}$$

$$\hat{R}^{DE}{}_{BA} = I\!R^{DE}{}_{BA} - 2\,I\!P_{a[B}{}^{D}\,I\!P^{a}{}_{A]}{}^{E} \quad . \tag{4.3.69}$$

Extensive use has been made of the generalised Gauss-Weingarten equations and the identities (4.3.47), (4.3.48), and (4.3.53). We have defined $I\!R^{DE}{}_{BA}$ by $I\!R^{DE}{}_{BA} = \gamma^{EF}\,I\!R^{D}{}_{FBA}$ where

$$I\!R^{D}{}_{EBA} = I\!\Gamma^{D}_{AE,B} - I\!\Gamma^{D}_{BE,A} + I\!\Gamma^{F}_{AE}\,I\!\Gamma^{D}_{BF} - I\!\Gamma^{F}_{BE}\,I\!\Gamma^{D}_{AF} \quad . \tag{4.3.70}$$

This curvature tensor is not derived from a Riemannian metric and it does not have the usual symmetries we associate with the standard Riemann tensor, except for $I\!R^{D}{}_{FBA} = -\,I\!R^{D}{}_{FAB}$ and the Bianchi identity $I\!R^{D}{}_{F[BA\|E]} = 0$. We are now able to derive the projected components of the Ricci tensor: $\hat{R}_{ab}$, $\hat{R}_{a}{}^{A}$, $\hat{R}^{AB}$. We have

$$\hat{R}_{ab} = g^{\gamma\delta}\,R_{\gamma\alpha\delta\beta}\,e^{\alpha}_{(a)}\,e^{\beta}_{(b)} = \gamma^{cd}\,\hat{R}_{acbd} + \gamma_{AB}\,\hat{R}_{a}{}^{A}{}_{b}{}^{B} \quad . \tag{4.3.71}$$

The last term in equation (4.3.71) can be further expanded by noting that equation (4.3.37) can be re-arranged to get

$$e^{(B)\alpha} = \gamma^{BA}\,e^{\alpha}_{(A)} - \gamma^{Ba}\,e^{\alpha}_{(a)} \quad . \tag{4.3.72}$$

The expansion results in the following equation:

$$\hat{R}_{ab} = \gamma^{cd}\,\hat{R}_{acbd} - \hat{R}^{A}{}_{abA} + \gamma^{c}_{A}\,\hat{R}^{A}{}_{abc} \quad . \tag{4.3.73}$$

We obtain similar equations for $\hat{R}_a{}^A$ and $\hat{R}^{AB}$,

$$\hat{R}_a{}^A = \hat{R}^{AB}{}_{aB} + \gamma^{cd}\hat{R}^A_{dac} + 2\gamma^{Bb}\hat{R}^A_{[ab]B} + \gamma^b_B\gamma^{Bc}\hat{R}^A{}_{cab} \quad (4.3.74)$$

and

$$\begin{aligned}\hat{R}^{AB} &= \gamma^{BE}\hat{R}^{AD}{}_{ED} - \gamma^{ab}\gamma^{BD}\hat{R}^A{}_{abD} - \gamma^{ab}\gamma^{Bc}\hat{R}^A{}_{acb} - \gamma^{Ba}\hat{R}^{AD}{}_{aD} \\ &+ \gamma^a_D\gamma^{BE}\hat{R}^{AD}{}_{aE} + 2\gamma^{Da}\gamma^{Bb}\hat{R}^A_{[ab]D} - \gamma^a_D\gamma^{Dc}\gamma^{Bb}\hat{R}^A{}_{cba} \quad . \quad (4.3.75)\end{aligned}$$

The complete form of these equations is obtained by using the Gauss-Codazzi equations. We have relegated the results to appendix D since some of the expressions are cumbersome.

4.4 Reduction to ADM

A worthwhile exercise is to see how our formalism relates to the standard ADM formalism. We shall consider a 3+1 generalised split. Comparing the line elements (4.2.7) and (4.3.41) we can directly read off that:

$$\gamma_{ab} \longrightarrow g_{ab} \quad (4.4.76)$$

$$\gamma^b_A \longrightarrow N^b \quad (4.4.77)$$

$$\gamma_{AB} \longrightarrow \epsilon N^2 \quad . \quad (4.4.78)$$

By comparing (4.2.6) and(4.3.37) we see that

$$e^{(B)\mu} \longrightarrow \epsilon N^{-1} n^\mu \quad . \quad (4.4.79)$$

It is obvious that $e^{(B)\mu}$ is not a unit vector; unlike n^μ. Examing the two sets of Gauss-Weingarten equations (4.2.12), (4.2.14), (4.2.19), (4.2.22), and (4.3.42),

(4.3.43), (4.3.49), (4.3.50), we are able to see that

$$K_{Aab} \longrightarrow N K_{ab} \tag{4.4.80}$$

$$P_{aB}{}^{A} \longrightarrow N^{-1} N_{,a} \tag{4.4.81}$$

$$\mathbb{K}_{Aab} \longrightarrow A_{ab} \tag{4.4.82}$$

$$\mathbb{P}_{aAB} \longrightarrow \epsilon N A_{a} \tag{4.4.83}$$

$$\mathbb{\Gamma}^{B}_{AD} \longrightarrow N^{-1} \dot{N} \ . \tag{4.4.84}$$

With the above results it is not too difficult to show that the generalised 3+1 formalism is equivalent to the ADM 3+1 formalism.

4.5 Orthogonal Spacetime Splitting

Among all the possible spacetime splittings a subset of particular interest is what we call "orthogonal" spacetime splitting. We define orthogonal spacetime splitting to mean that the horizontal spaces and the slanted spaces are globally orthogonal. Here the spacetime is $m+n$ decomposable; in general, this is not always feasible. In a coordinate specific form this statement translates into the following orthogonality condition: $e_{(A)} \cdot e_{(a)} = 0$. This condition is invariant under transformations of the two sets of intrinsic coordinates, that is, $\zeta^{A} \rightarrow \zeta^{A'}$, $\xi^{a} \rightarrow \xi^{a'}$. The orthogonality condition is most simply expressed as the fact that all the generalised shift vectors are zero, that is, $\gamma^{a}_{A} = 0$. These spacetime splitting are easily recognised since the line element looks like

$$ds^2 = \gamma_{ab}\, d\xi^{a} d\xi^{b} + \gamma_{AB}\, d\zeta^{A} d\zeta^{B} \ . \tag{4.5.85}$$

One of the first thing that we notice on examining (4.3.36) and (4.3.37) is that now

$$e^{(a)}_{\mu} = \gamma^{ab}\, e_{(b)\mu} \tag{4.5.86}$$

$$e^{\mu}_{(A)} \;=\; \gamma_{AB}\, e^{(B)\mu} \quad . \tag{4.5.87}$$

Therefore, the intrinsic metrics also raise and lower the labeling of the basis vectors.

We find $\mathbb{K}_{Aab}$ and $\mathbb{P}_{aAB}$ reduce to K_{Aab} and P_{aAB}, respectively. The generalised Gauss-Weingarten equations (4.3.42), (4.3.43), (4.3.49), and (4.3.50) now read as follows:

$$\frac{\delta e^{\alpha}_{(a)}}{\delta \xi^{b}} \;=\; -\,K_{Aab}\, e^{(A)\alpha} + \Gamma^{c}_{ab}\, e^{\alpha}_{(c)} \tag{4.5.88}$$

$$\frac{\delta e^{(A)\alpha}}{\delta \xi^{b}} \;=\; K^{A}{}_{b}{}^{c}\, e^{\alpha}_{(c)} - P_{bB}{}^{A}\, e^{(B)\alpha} \tag{4.5.89}$$

$$\frac{\delta e^{\alpha}_{(a)}}{\delta \zeta^{A}} \;=\; K_{A}{}^{b}{}_{a}\, e^{\alpha}_{(b)} + P_{aAB}\, e^{(B)\alpha} \tag{4.5.90}$$

$$\frac{\delta e^{(B)\alpha}}{\delta \zeta^{A}} \;=\; -\,P^{a}{}_{A}{}^{B}\, e^{\alpha}_{(a)} - \mathbb{\Gamma}^{B}_{AD}\, e^{(D)\alpha} \quad . \tag{4.5.91}$$

From the definition (4.3.56) we have

$$\begin{aligned} \mathbb{\Gamma}^{B}_{AD} \;&=\; -\frac{\delta e^{(B)}}{\delta \zeta^{A}} \cdot e_{(D)} = \frac{\delta e_{(D)}}{\delta \zeta^{A}} \cdot e^{(B)} = \frac{\delta e_{(A)}}{\delta \zeta^{D}} \cdot e^{(B)} \\ &=\; \mathbb{\Gamma}^{B}_{DA} \quad . \end{aligned} \tag{4.5.92}$$

By writing down the cyclic permutations of $\gamma_{AB\|D} = 0$ and using the symmetric property of $\mathbb{\Gamma}^{B}_{AD}$ we find

$$\mathbb{\Gamma}^{B}_{AD} \;=\; \frac{1}{2}\,\gamma^{BE}\left(\gamma_{EA,D} + \gamma_{ED,A} - \gamma_{AD,E}\right) \quad . \tag{4.5.93}$$

Therefore, $\mathbb{\Gamma}^{B}_{AD}$ is the Christoffel symbol associated with the metric γ_{AB}.

We can write

$$\begin{aligned} P_{bB}{}^{A} \;&=\; -\,e^{(A)}_{\alpha|\beta}\, e^{\beta}_{(b)}\, e^{\alpha}_{(B)} \;=\; -\,\frac{\delta e^{(A)}}{\delta \xi^{b}} \cdot e_{(B)} \\ &=\; \frac{\delta e_{(B)}}{\delta \xi^{b}} \cdot e^{(A)} \qquad = \; \frac{\delta e_{(a)}}{\delta \zeta^{B}} \cdot e^{(A)} \quad . \end{aligned} \tag{4.5.94}$$

It follows that $P_{aBA} = \frac{\delta e_{(a)}}{\delta \zeta^B} \cdot e_{(A)} = -\frac{\delta e_{(A)}}{\delta \zeta^B} \cdot e_{(a)} = P_{aAB}$. By virtue of this result and equation (4.3.48) we have

$$P_{aAB} = \frac{1}{2} \gamma_{AB,a} \quad . \tag{4.5.95}$$

The result of a similar analysis carried out on K_{Aab} is

$$K_{Aab} = \frac{1}{2} \gamma_{ab,A} \quad . \tag{4.5.96}$$

The generalised Gauss-Codazzi equations have an elegant symmetric form in the case of the orthogonal splitting,

$$\hat{R}_{abcd} = R_{abcd} + 2 K_{Ab[c} K^A{}_{d]a} \tag{4.5.97}$$

$$\hat{R}_{Aabc} = 2 K_{Aa[b;c]} - 2 K_{Ba[b} P_{c]A}{}^B \tag{4.5.98}$$

$$\hat{R}_{BbaA} = K_{Bab||A} - K_{Acb} K_{Ba}{}^c - P_{aDB} P_{bA}{}^D + P_{bAB;a} \tag{4.5.99}$$

$$\hat{R}_{aABC} = 2 P_{aA[B||C]} - 2 P^b{}_{A[B} K_{C]ba} \tag{4.5.100}$$

$$\hat{R}_{ABCD} = {}^{\mathbb{R}}R_{ABCD} + 2 P_{aB[C} P^a{}_{D]A} \quad . \tag{4.5.101}$$

The projected components of the Ricci tensor take on a much more manageable form, as compared to the generalised equations. The equations are endowed with an elegant symmetric appearance. The equations (D.1) and (D.3) reduce to

$$\begin{aligned} \hat{R}_{ab} = {} & R_{ab} - K^A K_{Aab} + 2 K_{Aa}{}^c K^A{}_{bc} \\ & - K^A{}_{ab||A} - P_{a;b} - P_{aAB} P_b{}^{AB} \end{aligned} \tag{4.5.102}$$

$$\begin{aligned} \hat{R}_{AB} = {} & {}^{\mathbb{R}}R_{AB} - P^a P_{aAB} + 2 P_{aA}{}^C P^a{}_{BC} \\ & - P^a{}_{AB;a} - K_{A||B} - K_{Aab} K_B{}^{ab} \quad . \end{aligned} \tag{4.5.103}$$

We note that $\hat{R}_{AB}$ has the same form as $\hat{R}_{ab}$; the upper and lower case roman indices have been interchanged, along with P_{aAB} and K_{Aab}. The mixed term $\hat{R}_{aA}$

does not at first sight look symmetric. It possesses an unsightly $\Gamma^{E}_{BD,a}$ term. At a first glance the equation (D.2) reduces to

$$\begin{aligned}\hat{R}_{aA} &= \gamma_{AB}\left(K^{Bb}{}_{;b} - K^{B}{}_{,a}\right) - K^{B}\,P_{aBA} \\ &\quad + \gamma_{AE}\,\gamma^{BD}\,\Gamma^{E}_{BD,a} - P_{aA}{}^{B}{}_{\|B} + K_{Aa}{}^{c}\,P^{c} \quad . \end{aligned} \tag{4.5.104}$$

The term that explicitly involves $\Gamma^{E}_{BD,a}$ can be replaced using the identity

$$\gamma_{AE}\,\Gamma^{E}_{BD,a} = P_{aDA\|B} + P_{aBA\|D} - P_{aBD\|A} \tag{4.5.105}$$

We note that this identity is a consequence of the orthogonal split and is not necessarily true in general. By using the equations (4.5.93) and (4.5.95) it can easily be shown that the left hand side of (4.5.105) is equivalent to the right hand side. The final form for $\hat{R}_{aA}$ is

$$\begin{aligned}\hat{R}_{aA} &= K_{Aa}{}^{b}{}_{;b} - K_{A,a} + K^{B}\,P_{aAB} - 2\,K^{B}_{ba}\,P^{b}_{BA} \\ &\quad + P_{aA}{}^{B}{}_{\|B} - P_{a,A} + P^{b}\,K_{Aab} \quad . \end{aligned} \tag{4.5.106}$$

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CHAPTER FIVE

GEODESIC PROPERTY OF COSMIC STRINGS

5.1 Introduction

Cosmic strings have long been present in mechanisms for forming large scale structure in the universe[1,2,3]. Infinitely long relativistic strings form an accretion wake which can help to explain the observed large-scale filaments. Vilenkin[4,5] has suggested that long-lived closed loops of string can act as seeds for galaxy and cluster formation. Grand unification strings can generate density fluctuations consistent with the requirements for galaxy formation.

Clarke, Ellis, and Vickers (CEV)[6] derived results which questioned the possibility of having closed loops on length scales small compared to the cosmological horizon. They assumed "the conical singularity model of cosmic strings to be approximately valid so that the region of high curvature may be confined to a narrow tube" and found that strings are very inflexible rigid objects. However, as they pointed out these finding may not necessarily be true if the conditions on the curvature are relaxed and the curvature is allowed to develop a singularity at the core.

CEV's claim seems surprising and appears to be in direct contradiction with explicit cosmic string solutions of Frolov, Israel, and Unruh (FIU)[7]. A family of momentarily stationary circular loops at a moment of time symmetry found by FIU contained closed loops of arbitrary loop radii and deficit angles.

In this chapter, we attempt to clear up the confusion surrounding closed

cosmic strings. We analyse idealised cosmic strings, investigating the behaviour of the Riemann curvature in the proximity of the string. We adopt the conical singularity model of cosmic strings where every point on the world sheet of the string is taken to be a conical singularity of the full spacetime. We refrain from imposing any conditions on the Riemann tensor but insist that the angular deficit will play a major role in determining the geometry of the spacetime in the immediate vicinity of the string. We find that the curvature diverges on the string for deficit angles less than π. However, this divergence turns out to be quite mild.

In section two, we also investigate the geodesic property of cosmic strings along with the behaviour of the curvature. We claim that intrinsic geodesics in the world sheet of the cosmic string are also global geodesics. This claim bears true for idealised infinitely thin cosmic strings. Indeed, the acceleration of a family of curves each at a fixed proper distance from the string falls to zero as the string is approached. The convergence to zero is so slow for grand unification strings that at the string's surface the acceleration is practically equal to its "conventional" background value and the equations of motion of the string are determined for all practical purposes by the usual Nambu action.

Section three contains a general description of the two classes of FIU circular cosmic strings. FIU examined two classes of circular loops of string at a moment of time symmetry. One class consists of a family of string solutions where the spacetime geometry changes abruptly from being conical in nature to being Schwarzschild-like. The other class consists of a family where the transition occurs smoothly. In the latter class, a segment of space that resides between two spherical caps spanning the loop is surgically removed and opposite faces are identified. The identification only makes sense if a conformal transformation is applied to the spacetime such that the conformal factor makes the space conformally flat and the

extrinsic curvature of the opposite faces zero.

An extensive study of the properties of the FIU solutions is undertaken in section four. We demonstrate that the two classes of strings have attributes identical with the general finding of section two.

We end the chapter with a reminder of the more important results.

5.2 General properties

There are two questions that we wish to answer in this section: (i) what is the general behaviour of the Riemann tensor in the vicinity of the string world sheet, and (ii) are cosmic strings geodesics of spacetime?

Before we analyse the problems at hand we need to set up some suitable coordinate system. We model the history of an idealised cosmic string as a two dimensional timelike submanifold, S_{zt}, whose points are conical singularities of spacetime.

We focus attention on a particular intrinsic geodesic, call it L_t, of S_{zt} and we choose the parameter t to be proper distance along L_t. The tangent vector to L_t is $u^\alpha = \frac{dx^\alpha}{dt}$. Let Q be any point in S_{zt} then there exists an intrinsic geodesic, L_z, through Q that intersects L_t orthogonally at the point R. We take z to be proper distance along L_z. There exists a two-dimensional spacelike conical subspace, S_{xy}, through Q that is swept out by geodesics of the spacetime orthogonal to S_{zt}. We can define Gaussian polar coordinates (ρ, ϕ) in S_{xy} with Q as the origin.

We can assign quasicylindrical coordinates (t, z, ρ, ϕ) to any point P sufficiently close to L_t. The point P lies on one of the S_{xy} subspaces which intersect S_{zt} at Q, say. Once Q is known we can determine the coordinates z and t in the

manner describe in the previous paragraph. See figure 5.1.

The first question we pose is: what is the behaviour of the Riemann tensor in the vicinity of an idealised cosmic string? We examine the hypersurface $t =$ *constant* and we assume that the extrinsic curvature is well-behaved and regular at $\rho = 0$. The restriction on the extrinsic curvature expresses the requirement that the conical structure at $\rho = 0$ is a property of spacetime and not an artifact of the embedding of a cone-like two-space in a regular space. The restriction is imposed so as to avoid any "bad slicing" of spacetime. Performing an ADM 3+1 split and choosing t as our ADM time, we find that the normal vector is $n_a \equiv u_a = \epsilon N \partial_a t$ where N is the lapse function, and $n.n = \epsilon$. The quantity ϵ takes values ± 1 according to whether the hypersurface being considered is timelike or spacelike.

Close to L_t, the four-metric takes the ADM form [see equation (4.2.7) in chapter four]

$$ds^2 = \epsilon N^2\, dt^2 + g_{ab}(dx^a + N^a\, dt)(dx^b + N^b\, dt) \quad . \qquad (5.2.1)$$

On L_t the lapse and shift are $N = 1$ and $N^a = 0$, respectively. For reasonably small values of ρ the string appears straight and the usual straight cosmic string metric[8] is a good approximation for describing the geometry of the string. Therefore, in the vicinity of L_t the spacetime metric is

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + (\frac{\rho}{\eta})^2\, d\phi^2 \qquad (0 \leq \phi < 2\pi) \qquad (5.2.2)$$

where η is related to the conical deficit angle, $\Delta\phi$, by

$$\frac{1}{\eta} = 1 - \frac{\Delta\phi}{2\pi} \quad . \qquad (5.2.3)$$

We consider the acceleration of a sequence of curves $\rho = \rho_0$ parallel to L_t.

The acceleration of each curve is given by

$$a_\alpha = u_{\alpha|\beta}\, u^\beta = \frac{\delta u_\alpha}{\delta t} = -\epsilon\,(N^{,a} - \epsilon\, K^{ab}\, N_b)\, e_{(a)\alpha} \tag{5.2.4}$$

which follows from equation (4.2.22) in chapter four. For ρ small

$$a_\alpha \simeq -\epsilon\, N^{,a}\, e_{(a)\alpha} = -\epsilon\, N^{,a}\, g_{\alpha\beta}\, e^\beta_{(a)} \simeq -\epsilon\, N_{,\alpha} \quad . \tag{5.2.5}$$

We now make a small digression, and examine the two-metric of the surface S_{xy}

$$ds^2 \;=\; d\rho^2 \,+\, (\frac{\rho}{\eta})^2\, d\phi^2 \quad . \tag{5.2.6}$$

This two-metric can be written in a conformally flat form, and once this has been done, we can speak about direction and angles at the conical vertex since angles are conformally invariant.

Formally, we express the coordinates (ρ, ϕ) in terms of quasi-Cartesian coordinates $x^A = x, y$ by $x = r\cos\phi$ and $y = r\sin\phi$ where

$$r \;=\; A(\frac{\rho}{A})^\eta \quad . \tag{5.2.7}$$

We have introduced A, which is some length scale associated with the string, so that the coordinates r and ρ both have units of length.

The two-metric of S_{xy} has a conformally flat form

$$ds^2 \;=\; d\rho^2 \,+\, (\frac{\rho}{\eta})^2\, d\phi^2 \;=\; \frac{1}{\eta^2}(\frac{r}{A})^{\frac{2(1-\eta)}{\eta}}\,(dx^2 \,+\, dy^2) \quad . \tag{5.2.8}$$

Returning to the problem at hand, the assumption that L_t is an intrinsic geodesic implies that the acceleration vector must lie in the two-surface S_{xy} which is perpendicular to S_{xt}. Thus, in the neighbourhood of L_t the acceleration has the asymptotic form $a_\alpha \approx \kappa_A(x^\beta)\,\delta^A_\alpha$ where $A = x, y$.

We require that the vector $\kappa_A(z^c)$ is well defined at the origin in the x, y coordinate system. Therefore, we demand that $\frac{\partial \kappa_A}{\partial \phi}|_{r=0}$. A direct consequence of this demand is that κ_A is independent of x and y at the origin. Therefore, we find that $a_\alpha(\rho = 0) = \kappa_A(z,t)\delta^A_\alpha$. We can now integrate equation (5.2.5) to get

$$-\epsilon N \simeq C + \kappa_A x^A \qquad (5.2.9)$$

where C is a constant of integration. By examining the metric (5.2.2) we find $N|_{\rho=0} = 1$. Therefore to first order the lapse is

$$N \simeq 1 - \epsilon \kappa_A x^A \quad . \qquad (5.2.10)$$

Having obtained the asymptotic form of the lapse function we are now ready to examine the behaviour of the Riemann tensor. We see from examining the Gauss-Weingarten equations[11] that the components which are the most divergent are

$$N \hat{R}_{\perp ab\perp} = \dot{K}_{ab} - K_{ab;c} N^c - 2N^c_{\ ;(a} K_{b)c} - N K_{ac} K_b^{\ c} + \overset{\circ}{\epsilon} N_{;ab} \quad . \qquad (5.2.11)$$

These equations reduce to

$${}^4R_{\perp ab\perp} = \epsilon N_{;ab} + \cdots \quad . \qquad (5.2.12)$$

where '$\cdots$' are terms that are regular in the limit ρ goes to zero. Now we have

$$N_{;ab} = N_{,ab} - N_{,c}\Gamma^c_{\ ab} \simeq \epsilon \kappa_C \Gamma^C_{\ ab} \quad . \qquad (5.2.13)$$

This in turn implies that

$${}^4R_{\perp ab\perp} = \kappa_C \Gamma^C_{\ ab} + \cdots \quad . \qquad (5.2.14)$$

In particular, the Christoffel symbol is

$$\Gamma^C_{\ AB} = \frac{1}{2} g^{CD}(g_{DA,B} + g_{DB,A} - g_{AB,D}) \quad . \qquad (5.2.15)$$

Remembering that

$$g_{DA} \simeq \frac{1}{\eta^2}\left(\frac{r}{A}\right)^{\frac{2(1-\eta)}{\eta}} \delta_{DA} \tag{5.2.16}$$

we find

$$g_{DA,B} \simeq \frac{1-\eta}{\eta}\frac{2x_B}{r^2} g_{DA} \tag{5.2.17}$$

with the notation, $x_B = \delta_{BC}\, x^C$. For small values of r we see

$$^4R_{\perp AB\perp} \simeq \frac{1-\eta}{\eta}\frac{1}{r^2}\{\kappa_A x_B + \kappa_B x_A - \delta_{AB}\kappa_C x^C\} + \cdots \quad . \tag{5.2.18}$$

We define a characteristic global length scale, a, associated with the loop by

$$\frac{1}{a} = \sqrt{\delta^{AB}\kappa_A\kappa_B} \quad . \tag{5.2.19}$$

Noting that $|\,\kappa_A\,| \simeq \frac{1}{a}$ and $|\,x_A\,| \simeq r$, we find

$$|\,^4R^{\perp A}{}_{B\perp}\,| \simeq \eta(\eta-1)\frac{1}{ar}\left(\frac{r}{A}\right)^{\frac{2(\eta-1)}{\eta}} \quad . \tag{5.2.20}$$

In terms of the physically relevant quantity, ρ, equation (5.2.20) reads

$$|\,^4R^{\perp A}{}_{B\perp}\,| \simeq \eta(\eta-1)\frac{1}{aA}\left(\frac{\rho}{A}\right)^{\eta-2} \quad . \tag{5.2.21}$$

We observe that

$$\eta - 2 = \frac{2(\Delta\phi - \pi)}{2\pi - \Delta\phi} \quad . \tag{5.2.22}$$

Thus $|\,^4R^{\perp A}{}_{B\perp}\,|$ diverges in the limit ρ goes to zero for $\Delta\phi < \pi$.

We are now ready to calculate the magnitude of the acceleration, which is a scalar given by

$$\kappa = \sqrt{|\,a_\alpha\, a^\alpha\,|} \simeq \sqrt{g^{\alpha\beta}\,\kappa_A\,\kappa_B\,\delta^A_\alpha\,\delta^B_\beta} \quad . \tag{5.2.23}$$

For the conical space S_{xy} we have

$$g^{AB} = \eta^2\left(\frac{r}{A}\right)^{\frac{2(\eta-1)}{\eta}} \delta^{AB} \tag{5.2.24}$$

so that

$$\kappa = \eta(\frac{r}{A})^{\frac{\eta-1}{\eta}}\sqrt{\delta^{AB}\kappa_A\kappa_B} \quad . \tag{5.2.25}$$

By virtue of equation (5.2.19), the magnitude of the acceleration is

$$\kappa = \frac{\eta}{a}(\frac{r}{A})^{\frac{\eta-1}{\eta}} \tag{5.2.26}$$

or converting back to proper distance from the string, ρ,

$$\text{magnitude of acceleration} \quad = \kappa = \frac{\eta}{a}(\frac{\rho}{A})^{\eta-1} \quad . \tag{5.2.27}$$

Our limiting form of the acceleration is a slight modification of the result of Hayward[9] and is the more precise form of the original result derived by Unruh, Hayward, Israel, and Mc Manus[10].

It is immediately apparent for an idealised cosmic string of zero thickness, that κ goes to zero in the limit ρ tends to zero. Therefore, an idealised cosmic string is indeed a geodesic of spacetime. We note that the divergence of the Riemann tensor for $\Delta\phi \leq \pi$ is weak enough to maintain the fact that L_t is still a global geodesic.

So far all our arguments have been local and we have made a number of assumptions, namely that the extrinsic curvature is regular, the conical nature of S_{xy}, etc.. We can check the validity of our results and assumptions by examining the exact initial-value solutions of FIU. (Note, we cannot check the validity of the assumption on the extrinsic curvature for the FIU solutions since these solutions only exist at a specific $t = const.$ The 4-geometry is not known so the extrinsic curvature of the 3-surface cannot be calculated.)

5.3 FIU circular cosmic strings

FIU studied a circular loop of string at a moment of time symmetry. The problem of finding a time-symmetric cosmic string was formulated as a boundary-value problem.

Two sets of solutions were considered: (i) where the transition of the geometry from a conical nature form to a Schwarzschild-type form occurs quite abruptly over a short distance, and (ii) where the transition is smooth.

We present a general description of the nature of these geometries before analysing specific examples. We impose axial symmetry. The three-metric of the time symmetric slice has the standard Weyl[12] form

$$ds^2 = e^{2(\nu-\lambda)}(d\rho^2 + dz^2) + \rho^2 e^{-2\lambda} d\phi^2 \qquad (5.3.28)$$

where λ, ν are functions of the variables ρ, z only. At time $t = 0$ the cosmic string is assumed to be momentarily at rest and have a circular configuration. The cosmic string is located on the ring $\rho = a, z = 0$. Several boundary conditions are imposed on the solution; the metric should be asymptotically flat and the axis of symmetry, $z = 0$, should be free of conical singularities.

For convenience, we transform to toroidal coordinates (σ, ψ) via

$$\rho = a N^{-2} \sinh\sigma \qquad (5.3.29)$$

$$z = a N^{-2} \sin\psi \qquad (5.3.30)$$

where $N = \sqrt{\cosh\sigma - \cos\psi}$. Consequently the three-metric has the form

$$ds^2 = a^2 N^{-4} e^{-2\lambda}[e^{2\nu}(d\sigma^2 + d\psi^2) + \sinh^2\sigma\, d\phi^2] \ . \qquad (5.3.31)$$

The spacetime in the vicinity of the string at $\sigma = \infty$ is assumed to be conical in nature which places constraints on the form of ν.

Firstly, we supply a brief description of the two classes of FIU cosmic string solutions.

a) Class 1: A particular class of circular string loops

At a given moment of time symmetry, the geometry of spacetime has the characteristic of being conical and locally flat in the immediate vicinity of the string loop. We pass through a transition zone as we move further away from the string. The geometry in the transition zone evolves from a conical form to a Schwarzschild-type form. Beyond the transition zone the space is free from gravitational radiation.

FIU idealised the transition zone as the thin wall of a toroidal tube enclosing the string loop. Consistent with this idealisation FIU put forward the following ansatz for the function ν,

$$\nu = \begin{cases} 4\alpha(\sigma - \sigma_0) & (\sigma > \sigma_0) \\ 0 & (\sigma < \sigma_0) \end{cases} \tag{5.3.32}$$

where α is a measure of the deficit angle, $\alpha = \frac{\Delta\phi}{8\pi}$, and the toroidal tube is situated at $\sigma = \sigma_0 = constant$.

FIU introduced a function V which is directly related to λ by

$$V = V(\sigma, \psi) = e^{-\lambda/2} - 1 \tag{5.3.33}$$

and determined V explicitly to be

$$V = \frac{\sqrt{2}}{\pi} N \sum_{n=0}^{\infty} \frac{\epsilon_n \alpha b_n}{1 - \alpha b_n} \frac{P_{n-\frac{1}{2}}(\cosh\sigma_<)}{P_{n-\frac{1}{2}}(\cosh\sigma_0)} Q_{n-\frac{1}{2}}(\cosh\sigma_>) \cos n\psi \tag{5.3.34}$$

where the coefficients b_n and ϵ_n are

$$b_n = \sinh\sigma_0 \, P_{n-\frac{1}{2}}(\cosh\sigma_0) \, Q_{n-\frac{1}{2}}(\cosh\sigma_0) \tag{5.3.35}$$

$$\epsilon_n \;=\; 2 - \delta^0_n \quad . \tag{5.3.36}$$

$P_{n-\frac{1}{2}}$ and $Q_{n-\frac{1}{2}}$ are toroidal Legendre functions[13] and the notation $\sigma_< = \min\{\sigma, \sigma_0\}$, $\sigma_> = \max\{\sigma, \sigma_0\}$ has been introduced. Equations (5.3.33 – 5.3.36) determine λ in the metric (5.3.31) and in conjunction with the ansatz (5.3.32) completely specify a family of cosmic string solutions which we refer to as class one solutions. It can be shown that a solution exists for all $\Delta\phi$ if the torus $\sigma = \sigma_0$ on which the deficit is healed has $\sigma_0 < 2.625$.

b) Class 2: Conformally flat time-symmetric circular loops

The class one solutions have the unnatural property that the conical nature of spacetime is healed rather abruptly on a toroidal surface. A solution were the healing occurs smoothly was found by FIU. A pair of spherical caps spanning the loop were removed. At first sight the opposite spherical faces, which have the same intrinsic geometry, cannot be identified since their extrinsic curvatures are of opposite signs. The caps can be matched by applying a conformal transformation which (i) reduces to the identity at infinity, and (ii) flattens the extrinsic curvatures of the faces. FIU observed that the conformal factor is uniquely specified if the boundary conditions (i) and (ii) are satisfied, and if the conformally flat space is required to be empty.

The metric of a Euclidean flat space in three dimensions is

$$ds^2_{flat} \;=\; N^{-4}\, ds^2_* \tag{5.3.37}$$

where

$$ds^2_* \;=\; a^2\,(d\sigma^2 \;+\; d\psi^2 \;+\; \sinh^2\sigma\, d\phi^2) \quad . \tag{5.3.38}$$

Introducing a conformal factor $(\omega N)^4$ the physical metric is

$$ds^2 = \omega^4 \, ds_*^2 \quad . \tag{5.3.39}$$

The boundary conditions are satisfied by

$$\omega = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \epsilon_n \, Q_{n\eta-\frac{1}{2}}(\cosh\sigma) \, \cos(n\eta\psi) \tag{5.3.40}$$

where $Q_{n\eta-\frac{1}{2}}$ is a Legendre function of the second kind. The function ω can be expressed in an integral form as

$$\omega = \frac{1}{\pi} \int_{\sigma}^{\infty} \frac{d\tau}{\sqrt{\cosh\tau - \cosh\sigma}} \, \frac{\sinh\eta\tau}{\cosh\eta\tau - \cos\eta\psi} \quad . \tag{5.3.41}$$

Equations (5.3.38 – 5.3.41) determine the metric for a conformally flat string loop.

5.4 Riemann tensor calculations

We wish to investigate whether the FIU solutions have the general properties of cosmic strings which we derived in section two of this chapter. Our primary goal is to determine the form of the Riemann tensor near the cosmic string. We start by rewriting the three-metric (5.3.31) as

$$ds^2 = a^2 \, \omega^4 \, (d\sigma^2 + d\psi^2 + e^{-2\nu} \sinh^2\sigma \, d\phi^2) \tag{5.4.42}$$

where ν, ω are functions of the variables σ and ψ only. The above form of the metric is ideal for using the differential techniques of exterior calculus[14,15,16]. We write the metric in terms of differential 1-forms, θ^a, as

$$ds^2 = (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 \tag{5.4.43}$$

$$= g_{ab} \, \theta^a \, \theta^b \tag{5.4.44}$$

where $g_{ab} = \delta_{ab}$ and

$$\theta^1 = a\,\omega^2\,d\sigma \tag{5.4.45}$$

$$\theta^2 = a\,\omega^2\,d\psi \tag{5.4.46}$$

$$\theta^3 = a\,\omega^2\,e^{-\nu}\,\sinh\sigma\,d\phi \quad . \tag{5.4.47}$$

Cartan's first structural equation

$$d\theta^a = -\omega^a{}_b \wedge \theta^b \tag{5.4.48}$$

enables us to figure out the connection 1-forms $\omega^a{}_b$. The identity $dg_{ab} = \omega_{ab} + \omega_{ba}$ indicates that ω_{ab} is anti-symmetric. The non-zero connection 1-forms are

$$\omega^1{}_2 = \frac{2\omega_\psi}{a\,\omega^3}\,\theta^1 - \frac{2\omega_\sigma}{a\,\omega^3}\,\theta^2 = -\omega^2{}_1 \tag{5.4.49}$$

$$\omega^3{}_1 = \{\frac{2\omega_\sigma}{a\,\omega^3} + \frac{1}{a\,\omega^2}\,\frac{\cosh\sigma}{\sinh\sigma} - \frac{\nu_\sigma}{a\,\omega^2}\}\,\theta^3 = -\omega^1{}_3 \tag{5.4.50}$$

$$\omega^3{}_2 = \{\frac{2\omega_\psi}{a\,\omega^3} - \frac{\nu_\psi}{a\,\omega^2}\}\,\theta^3 = -\omega^2{}_3 \quad . \tag{5.4.51}$$

The curvature 2-form, $\Omega^a{}_b$, is given by Cartan's second structural equation

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad . \tag{5.4.52}$$

However, the curvature 2-form is defined in terms of the Riemann tensor by

$$\Omega^a{}_b = \frac{1}{2}\,R^a{}_{bcd}\,\theta^c \wedge \theta^d \quad . \tag{5.4.53}$$

We are thus able to simply read of the frame components of the Riemann tensor from equation (5.4.52). The non zero components of the Riemann tensor up to symmetries are

$$R_{1221} = \frac{2}{a^2\,\omega^6}\,\{(\omega_{\psi\psi} + \omega_{\sigma\sigma})\,\omega - \omega_\psi^2 - \omega_\sigma^2\} \tag{5.4.54}$$

$$R_{3113} = \frac{1}{a^2\,\omega^6}\,\{2\omega_{\sigma\sigma}\,\omega - 2\omega_\sigma^2 + 4\omega_\psi^2 + 2\omega_\sigma\,\omega\,\frac{\cosh\sigma}{\sinh\sigma} + \omega^2$$

$$-\omega^2\,\nu_{\sigma\sigma} - 2\omega\,\omega_\sigma\,\nu_\psi - 2\omega\,\omega_\psi\,\nu_\psi - 2\omega^2\,\nu_\sigma\,\frac{\cosh\sigma}{\sinh\sigma} + \omega^2\,\nu_\sigma^2\}\,(5.4.55)$$

$$R_{3123} = \frac{1}{a^2\,\omega^6}\,\{2\omega_{\sigma\psi}\,\omega - 6\omega_\sigma\omega_\psi - \omega^2\,\nu_{\sigma\nu} - \omega^2\nu_\psi\frac{\cosh\sigma}{\sinh\sigma} + \omega^2\,\nu_\psi\,\nu_\sigma\} \qquad (5.4.56)$$

$$R_{3223} = \frac{2}{a^2\,\omega^6}\,\{\omega_{\psi\psi}\,\omega - \omega_\psi^2 + 2\omega_\sigma^2 + \omega\,\omega_\sigma\frac{\cosh\sigma}{\sinh\sigma} - \frac{1}{2}\,\omega^2\,\nu_{\psi\psi}\omega\,\omega_\psi\,\nu_\psi - \omega\,\omega_\sigma\,\nu_\sigma + \frac{1}{2}\,\omega^2\,\nu_\psi^2\}\ . \qquad (5.4.57)$$

a) Class 2 solutions

We tackle the simpler of the two classes of FIU solutions first. For class two solutions we have $\nu = 0$. The integral expression for ω, (5.3.41), can be written in terms of exponentials as

$$\omega = \frac{\sqrt{2}}{\pi}\int_\sigma^\infty \frac{d\tau}{\sqrt{e^\tau + e^{-\tau} - e^\sigma - e^{-\sigma}}}\,\frac{e^{\eta\tau} - e^{-\eta\tau}}{e^{\eta\tau} + e^{-\eta\tau} - 2\cos\eta\tau}\ . \qquad (5.4.58)$$

We observe that

$$e^\tau + e^{-\tau} - e^\sigma - e^{-\sigma} = (e^\tau - e^\sigma)\,\{1 - e^{-(\tau+\sigma)}\} \qquad (5.4.59)$$

and consider the case $\tau \geq \sigma \gg 1$. We assume that an angular deficit exists, that is $\eta > 1$. The only terms in ω which are of any importance in the given limit are

$$\omega = \frac{\sqrt{2}}{\pi}\int_\sigma^\infty \frac{d\tau}{\sqrt{e^\tau - e^\sigma}}\,\{1 + \frac{1}{2}\,e^{-(\tau+\sigma)} + 2\cos\eta\psi\,e^{-\eta\tau} + \cdots\} \qquad (5.4.60)$$

If we make the change of variable, $\sin x = e^{(\sigma-\tau)/2}$, then the integral for ω becomes

$$\omega = \frac{2\sqrt{2}}{\pi}\,e^{-\frac{\sigma}{2}}\int_0^{\frac{\pi}{2}} dx\{1 + \frac{1}{4}\,e^{-2\sigma}\,(1 - \cos 2x) + 2\cos\eta\psi\,e^{-\eta\sigma}\,\sin^{2\eta}x + \cdots\}\ . \qquad (5.4.61)$$

Evaluating this integral we find

$$\omega = \sqrt{2}\, e^{-\sigma/2} + B \cos\eta\psi\, e^{-(\eta+\frac{1}{2})\sigma} + \frac{\sqrt{2}}{4} e^{-3\sigma/2} + \cdots \tag{5.4.62}$$

where

$$B = B(\eta) = \frac{4\sqrt{2}}{\pi} \int_0^{\frac{\pi}{2}} dx\, (\sin^2 x)^\eta \leq 2\sqrt{2} \ . \tag{5.4.63}$$

We use equations (5.4.54 - 5.4.57) along with the asymptotic form of ω, (5.4.62), to write down the frame components of the Riemann tensor. Close to the string the Riemann tensor components are approximately

$$R_{1221} \simeq \frac{1}{2a^2} \tag{5.4.64}$$

$$R_{1313} \simeq \frac{1}{4a^2} - \frac{\sqrt{2}}{4a^2}\eta(\eta-1)\, B \cos\eta\psi\, e^{(2-\eta)\sigma} \tag{5.4.65}$$

$$R_{1332} \simeq \frac{\sqrt{2}}{4a^2}\eta(\eta-1)\, B \sin\eta\psi\, e^{(2-\eta)\sigma} \tag{5.4.66}$$

$$R_{2323} \simeq \frac{1}{4a^2} + \frac{\sqrt{2}}{4a^2}\eta(\eta-1)\, B \cos\eta\psi\, e^{(2-\eta)\sigma} \ . \tag{5.4.67}$$

We wish to compare these results to our findings in section two. We need to introduce a characteristic distance which we take to be the proper distance to the string

$$\rho = \int_\infty^\sigma d\sigma\, a\omega^2 \ . \tag{5.4.68}$$

For large values of σ we find

$$\rho \simeq 2ae^{-\sigma} \ . \tag{5.4.69}$$

The general behaviour of the Riemann tensor close to the string is

$$|\, R_{\ldots}\, | = \frac{\sqrt{2}}{4a^2}\eta(\eta-1)\, B\, e^{(2-\eta)\sigma} + \text{finite terms} \tag{5.4.70}$$

$$= \frac{\sqrt{2}}{4a^2}\eta(\eta-1)\, B\, (\frac{\rho}{2a})^{\eta-2} + \text{finite terms} \ . \tag{5.4.71}$$

We divert our attention to the geodesic nature of the cosmic string loop. We consider the acceleration of a family of concentric loops located at $\sigma = constant, \psi = constant$. On each of these loops we have

$$ds^2 = a^2\omega^4 \sinh^2\sigma\, e^{-2\nu}\, d\phi^2 \tag{5.4.72}$$

which implies

$$\frac{d\phi}{ds} = \frac{e^{\nu}}{a\omega^2 \sinh\sigma} \quad . \tag{5.4.73}$$

The components of the velocity vector are

$$u^{\alpha} = \frac{dx^{\alpha}}{ds} = \frac{d\phi}{ds}\,\delta^{\alpha}_{\phi} \quad . \tag{5.4.74}$$

We recall that the acceleration vector is

$$a^{\alpha} = u^{\alpha}|_{\beta}\, u^{\beta} = \Gamma^{\alpha}_{\phi\phi}\,(\frac{d\phi}{ds})^2 \tag{5.4.75}$$

where

$$\Gamma^{\alpha}_{\phi\phi} = -\frac{1}{2}\, g^{\alpha\beta}\, g_{\phi\phi,\beta} \quad . \tag{5.4.76}$$

We calculate the components of the acceleration vector using (5.4.75) and for loops characterised by a large value of σ we find that the non zero components are

$$a^{\psi} \simeq \frac{\sqrt{2}B\eta}{4a^2}\, e^{(2-\eta)\sigma}\, \sin\eta\psi \tag{5.4.77}$$

$$a^{\sigma} \simeq \frac{\sqrt{2}B\eta}{4a^2}\, e^{(2-\eta)\sigma}\, \cos\eta\psi \quad . \tag{5.4.78}$$

The magnitude of the acceleration is

$$\kappa \simeq a\omega^2\, \frac{\sqrt{2}B\eta}{4a^2}\, e^{(2-\eta)\sigma} \tag{5.4.79}$$

$$\simeq \frac{\sqrt{2}B\eta}{2a}\,(\frac{\rho}{2a})^{\eta-1} \tag{5.4.80}$$

and is in concordance with the general behaviour predicted by equation (5.2.27) where $B = \sqrt{2}(\frac{A}{2a})^{1-\eta}$. Replacing B in equation (5.4.71) by this expression, we find

$$|\,{}^4R_{\dots}\,| \simeq \eta(\eta-1)\,\frac{1}{aA}\,(\frac{\rho}{A})^{\eta-2} \quad , \tag{5.4.81}$$

which is in total agreement with the general behaviour predicted by equation (5.2.21).

b) Class 1 solutions

We are again interested in the asymptotic behaviour of the frame components of the Riemann tensor and the magnitude of the acceleration of concentric loops. We begin by examining the form of the three-metric in the regime where σ is quite large. Comparing the two forms of the metric (5.3.31) and (5.4.42) we see that they are related by

$$\omega = N^{-1}\,e^{(\nu-\lambda)/2} \quad . \tag{5.4.82}$$

We deduce with the aid of equation (5.3.33) that the form of the function ω for class one cosmic string solutions is

$$\omega = N^{-1}\,(1+V)\,e^{2\alpha(\sigma-\sigma_0)} \tag{5.4.83}$$

with V specified by equation (5.3.34). We can assume that for large values of σ we have $\sigma > \sigma_0$ and consequently

$$V = \frac{\sqrt{2}}{\pi}\,N \sum_{n=0}^{\infty} \frac{\epsilon_n \alpha b_n}{1-\alpha b_n}\,Q_{n-\frac{1}{2}}(\cosh\sigma)\,\cos n\psi \quad . \tag{5.4.84}$$

We obtain an asymptotic expression for $Q_{n-\frac{1}{2}}(\cosh\sigma)$ by making use of the integral expression[17]

$$Q_{n-\frac{1}{2}}(\cosh\sigma) = \frac{1}{\sqrt{2}} \int_0^{\pi} d\theta\, \frac{\cos n\theta}{\sqrt{\cosh\sigma - \cos\theta}} \quad . \tag{5.4.85}$$

Writing $\cosh\sigma$ in terms of exponentials and expanding in powers of $e^{-\sigma}$ we have

$$Q_{n-\frac{1}{2}}(\cosh\sigma) = e^{\frac{-\sigma}{2}} \int_0^\pi d\theta \, \cos n\theta (1 + e^{-\sigma} \cos\theta + \cdots) \quad . \tag{5.4.86}$$

Therefore for large values of σ

$$Q_{n-\frac{1}{2}}(\cosh\sigma) = \pi e^{-\frac{\sigma}{2}} (\delta_n^0 + \frac{1}{2} e^{-\sigma} \delta_n^1) + \mathcal{O}(e^{-\frac{3\sigma}{2}}) \quad . \tag{5.4.87}$$

Inserting equation (5.4.87) into equation (5.4.84) we discover that the asymptotic form of V is

$$V = \frac{\alpha b_0}{1 - \alpha b_0} + (\frac{\alpha b_1}{1 - \alpha b_1} - \frac{\alpha b_0}{1 - \alpha b_0}) e^{-\sigma} \cos\psi + \mathcal{O}(e^{-2\sigma}) \quad . \tag{5.4.88}$$

Hence, the asymptotic form of ω as given by equation (5.4.83) is

$$\omega = e^{-\frac{\sigma}{2}} D (1 + F e^{-\sigma} \cos\psi + \cdots) e^{2\alpha\sigma} \tag{5.4.89}$$

where D and F are constants whose values are determined by

$$D = \frac{\sqrt{2}}{1 - \alpha b_0} e^{-2\alpha\sigma_0} \tag{5.4.90}$$

$$F = \frac{1 - \alpha b_0}{1 - \alpha b_1} \tag{5.4.91}$$

The asymptotic frame components of the Riemann tensor follow directly from equations (5.4.54 – 5.4.57) with ν and ω specified by equations (5.3.32) and (5.4.89), respectively,

$$R_{1221} = 0 + \cdots \tag{5.4.92}$$

$$R_{3113} = \frac{8\alpha F}{a^2 D^4} e^{(1-8\alpha)\sigma} \cos\psi + \cdots \tag{5.4.93}$$

$$R_{3123} = \frac{8\alpha F}{a^2 D^4} e^{(1-8\alpha)\sigma} \sin\psi + \cdots \tag{5.4.94}$$

$$R_{3232} = \frac{8\alpha F}{a^2 D^4} e^{(1-8\alpha)\sigma} \cos\psi + \cdots \quad . \tag{5.4.95}$$

The proper distance from the cosmic string to a concentric loop is [see equation (5.4.68)]

$$\rho = aD^2 \int_{\infty}^{\sigma} d\sigma \, e^{(4\alpha-1)\sigma} \tag{5.4.96}$$

$$\simeq \frac{aD^2}{4\alpha-1} e^{(4\alpha-1)\sigma} \quad .$$

Consequently, the frame components of the Riemann tensor close to the string look like

$$| R_{\dots} | \simeq \frac{8\alpha F}{a^2 D^4} \left(\frac{4\alpha-1}{D^2} \frac{\rho}{a}\right)^{\frac{1-8\alpha}{4\alpha-1}} \tag{5.4.97}$$

and knowing that $8\alpha = \frac{\Delta\phi}{\pi} = 2(1-\frac{1}{\eta})$ enables us to reduce the above expression to

$$| R_{\dots} | \simeq \frac{2}{a^2} \frac{F}{D^4} (4\alpha-1)^{\eta} \eta(\eta-1) \left(\frac{1}{D^2} \frac{\rho}{a}\right)^{\eta-2} \quad . \tag{5.4.98}$$

The components of the acceleration vector of a loop situated reasonably close to the string behave like

$$a^{\psi} \simeq \frac{2F}{a^2 D^4} e^{(1-8\alpha)\sigma} \sin\psi \tag{5.4.99}$$

$$a^{\sigma} \simeq \frac{2F}{a^2 D^4} e^{(1-8\alpha)\sigma} \cos\psi \quad . \tag{5.4.100}$$

The magnitude of the acceleration is then

$$\kappa \simeq \frac{1}{a} \frac{2F}{D^2} e^{-4\alpha\sigma} \quad . \tag{5.4.101}$$

Re-expressing equation (5.4.101) in terms of ρ we find

$$\kappa \simeq \frac{1}{a} \frac{2F}{D^2} \left(\frac{4\alpha-1}{D^2} \frac{\rho}{a}\right)^{\frac{4\alpha}{1-4\alpha}} \tag{5.4.102}$$

$$\simeq \frac{\eta}{a} \frac{2F}{D^2} (4\alpha-1)^{\eta} \left(\frac{1}{D^2} \frac{\rho}{a}\right)^{\eta-1} \tag{5.4.103}$$

which is in agreement with the general result (5.2.27) where A is given by

$$A^{1-\eta} = \frac{2F}{D^2} (4\alpha-1)^{\eta} (aD^2)^{1-\eta} \quad . \tag{5.4.104}$$

With equation (5.4.104) we can show that the Riemann tensor behaviour given by the expression (5.4.98) is again identical to the general result predicted by equation (5.2.21).

5.5 Conclusion

The main results that we wish to emphasise concern the acceleration of curves close to the string and the limiting form of the Riemann tensor. The acceleration of a curve that is a fixed proper distance, ρ, away from the string depends on the deficit angle of the string according to

$$\kappa \sim \frac{\eta}{a}\left(\frac{\rho}{A}\right)^{\eta-1} \tag{5.5.105}$$

where a is a global characteristic length scale associated with the string, and A is a length scale that has the general form $A = a\,f(\eta)$. The limiting value of κ for an idealised cosmic string is zero as long as the deficit angle is positive. This ensures that such strings are geodesics of spacetime. However for grand unification strings we note $\eta - 1 = 10^{-6}$ so that the convergence to zero is so slow that the acceleration at the string surface to extremely high accuracy is practically equal to its "conventional" background value, $\frac{1}{a}$.

Close to an idealised cosmic string the four-curvature behaves as

$$|\,{}^4R_{\ldots}\,| \sim \eta(\eta-1)\frac{1}{aA}\left(\frac{\rho}{A}\right)^{\eta-2} \quad . \tag{5.5.106}$$

Therefore, the curvature converges for deficit angles greater than π but diverges for $\Delta\phi < \pi$. The divergence is weak enough that the Riemann tensor is integrable, and the metric and affine connection remain bounded and well defined at the string.

The two different classes of FIU string solutions that we examined both have an asymptotic behaviour that is identical to the general results (5.5.105) and (5.5.106).

And finally, CEV's result is essentially correct in stating that cosmic strings are "straight". However, we interpret "straight" in the geodesic context. Our results can best be summed up in the statement that idealised cosmic strings never bend: they only bend the space around them.

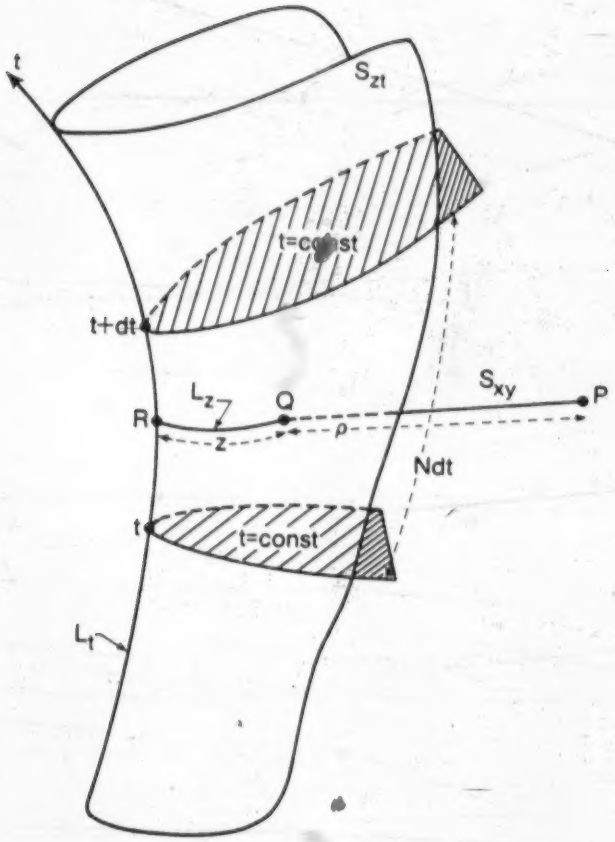

Figure 5.1: A three-section $\phi = const$ of spacetime, showing the two dimensional surfaces S_{zt} of a thin string. The conical two-surfaces S_{xy} perpendicular to the string at a given moment appear here as lines. If L_t (an intrinsic geodesic of S_{zt}) is not a geodesic of spacetime, its curvature will force neighbouring hypersurfaces $t = const$ to tilt relative to each other.

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CHAPTER SIX

CONCLUSION

In chapters two and three, we revisited two classical problems of general relativity; what matter configurations give rise to the Kerr[1] and the Kerr-Newman[2] black hole geometries? Many people have confronted these problems but no satisfactory concensus has been reached. We examined these problems using the techniques of surface layers. The proposed sources for the Kerr and the Kerr-Newman black hole geometries consisted of a toroidal shell that enclosed the ring singularity, a disc that spanned the equatorial plane, and an annulus of matter interior to the toroidal shell. We obtained the individual contributions to the mass and the angular momentum from the various parts that made up the source. In the case of the Kerr-Newman source, the electric charge was also calculated. In both cases, the sources had the correct mass and angular momentum. The Kerr-Newman source had the correct electric charge.

Indeed, the findings were similar to those of López[3,4] who employed the theory of distributions to obtain his results. López was under the impression that the theory of surface layers did not give the correct answer. However, the theories of surface layers and distributions are essentially the same when it comes to analysing the properties of hypersurfaces.

The remarkable feature of our results is that they bear out the original findings of Janis and Newman[5] who in 1965 suggested that the source of the Kerr black hole geometry consisted of a ring of matter together with a disc of matter spread over the equatorial plane.

In chapter four, we extended the ADM[6] 3+1 spacetime split to an arbitrary split of spacetime. We found that the concept of a lapse function and a shift vector needed to be generalised to a lapse matrix and a set of n shift vectors. Each m-dimensional foliating space had n independent orthogonal vectors and we associated n distinct generalised extrinsic curvature tensors with each foliating space. We derived the generalised Gauss-Weingarten equations and the generalised Gauss-Codazzi equations associated with the splitting. The projected components of the Ricci tensor were also calculated. The generalised formalism differs from the ADM formalism in that it does not demand the orthogonal vectors to be of unit length. As a result, a few extra terms appear in the generalised formalism.

The properties of submanifolds are well understood in the context of pure mathematics. However, the mathematical language of submanifolds is not readily accessible to physicists. The generalised formalism provides a workable technique that should be useful to a broad class of physicists, especially those already familiar with the ADM formalism. The applications of our formalism are widespread, such as cosmic strings, p-branes, and Kaluza-Klein theories.

In chapter five, we examined the properties of idealised cosmic strings. We modeled the world-sheet of an idealised cosmic string as a two dimensional surface whose points were conical singularities of spacetime. We obtained two general results: (i) idealised cosmic strings are geodesics of spacetime, (ii) the magnitude of the Riemann tensor diverges for conical deficit angles less than π. The former result was proved by examining the acceleration of a family of curves each at a fixed distance from the string. We found that the acceleration depended on the deficit angle and that for all positive deficit angles the magnitude of the acceleration went to zero as the proper distance of the curve from the string went to zero.

For grand unified strings, the deficit angle is extremely small and the acceleration at the string's surface is for most practical purposes equal to the conventional background value. This result holds to an exceedingly high accuracy. In the latter general result, the divergence of the Riemann tensor was weak enough that the Riemann tensor remained integrable, and that the metric and affine connection remained bounded and well defined at the string.

We drew the conclusion that the only constraint placed on the allowable configurations of idealised cosmic strings is that they must be geodesics of spacetime. Indeed, cosmic strings bend the space around them so that they are geodesics of spacetime. Our results indicate that closed cosmic string loops are allowable on all scales.

Cosmic strings have been implemented into two different proposals for galaxy formation: (i) accretion onto closed loops[7,8], and (ii) planar wake formation behind cosmic strings[9]. Originally when Clarke, Ellis, and Vickers (CEV)[10] presented their paper, the former proposal was the most fashionable. Thus CEV ruled out cosmic strings as having a role in galaxy formation. The role of cosmic strings in structure formation has since been completely revised. The latter mechanism is now considered the most prevalent. However, isolated closed loops can still play a role in galaxy formation, albeit a lesser one. Thus proposals of galaxy formation where matter is accreted onto loops cannot be dismissed just yet. Our results keep this option open.

We supported our general results by examining the properties of two particular sets of cosmic string solutions found by Frolov, Israel, and Unruh[11]. We found that the FIU string solutions had properties identical with our general findings.

We hope to apply the techniques learnt in this thesis to other problems. Now that we have a feel for the properties of hypersurfaces, future work should include examining the properties of null surfaces with the aim of finding a sensible definition for the Ricci tensor of a null surface. Another possible future project is to re-examine the properties of cosmic strings using the techniques of generalised spacetime splitting.

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APPENDIX A

KERR-SCHILD METRIC

The properties of Kerr-Schild[1] metrics are used quite extensively in chapters two and three. A Kerr-Schild metric is a metric that can be written in the following form

$$ds^2 = g_{ij}\,dx^i\,dx^j \quad \text{with} \quad g_{ij} = \eta_{ij} + 2H\,v_i v_j \tag{A.1}$$

where η_{ij} is the metric of Minkowski (flat) spacetime, H is an arbitrary scalar, and v_i is a null vector; it is null with respect to both the full metric, g_{ij} and the flat metric, η_{ij}, since $g_{ij}v^j = \eta_{ij}v^j$ (because $v_i v^i = 0$). We note that both the Kerr[2] metric for a rotating black hole and the Kerr-Newman[3] metric for a rotating charged black hole are Kerr-Schild metrics.

Since we deal with different Kerr-Schild metrics it is handy to first work with the general form of this type of metric. The contravariant components of the metric are

$$g^{ij} = \eta^{ij} - 2H\,v^i v^j \quad . \tag{A.2}$$

We shall raise and lower indices using the flat metric, η_{ij}. Another useful property of the metric is that $\det g_{ij} = \det \eta_{ij}$. Extending the work of Misra[4], the

Christoffel symbols for the metric are

$$\begin{aligned}
\Gamma^{l}_{ij} &= {}^{0}\Gamma^{l}_{ij} - 2H\,\eta_{pq}\,{}^{0}\Gamma^{q}_{ij}\,v^{p}v^{l} \\
&\quad + v^{l}[H_i v_j + H_j v_i + H(v_{i,j} + v_{j,i}) + 2H\,H_k v^{k} v_i v_j \\
&\quad + 2H^{2} v^{k}\{v_i(v_{j,k} - v_{k,j}) + v_j(v_{i,k} - v_{k,i})\}] \\
&\quad + \eta^{lk}[H\,v_i(v_{k,j} - v_{j,k}) + H\,v_j(v_{k,i} - v_{i,k}) - H_k\,v_i v_j]
\end{aligned} \tag{A.3}$$

where ${}^{0}\Gamma^{i}_{jk}$ are the Christoffel symbols for the metric η_{ij}. We note that

$$\Gamma^{k}_{ij}\,v_k v^{j} = {}^{0}\Gamma^{k}_{ij}\,v_k v^{j} \quad . \tag{A.4}$$

This implies that

$$v_{i|j}v^{j} = v_{i;j}v^{j} \tag{A.5}$$

where ";" and "|" denote covariant differentiation with respect to the metrics g_{ij} and η_{ij}, respectively.

We shall only consider Kerr-Schild metrics that have the geodesic property, $v_{i|j}\,v^{j} = 0$. The Kerr metric and the Kerr-Newman metric both satisfy this criterion. The Christoffel symbols then reduce to

$$\begin{aligned}
\Gamma^{l}_{ij} &= {}^{0}\Gamma^{l}_{ij} - 2H\,\eta_{pq}\,{}^{0}\Gamma^{q}_{ij}\,v^{p}v^{l} \\
&\quad + v^{l}\{H_i v_j + H_j v_i + H(v_{i,j} + v_{j,i}) + 2H\,H_k v^{k} v_i v_j\} \\
&\quad + \eta^{lk}\{H\,v_i(v_{k,j} - v_{j,k}) + H\,v_j(v_{k,i} - v_{i,k}) - H_k\,v_i v_j\}
\end{aligned} \quad . \tag{A.6}$$

The expression for the Ricci tensor is

$$R_{ij} = \Gamma^{k}_{il}\Gamma^{l}_{kj} - \Gamma^{k}_{ij}\Gamma^{l}_{kl} + \Gamma^{k}_{ik,j} - \Gamma^{k}_{ij,k} \quad . \tag{A.7}$$

In our case this reads

$$R_{ij} = \Box(H\, v_i v_j) - (b\, v_i)_{;j} - (b\, v_j)_{;i} + K\, v_i v_j \tag{A.8}$$

$$\text{with} \begin{cases} b &= H_k v^k + H\, v^k_{;k} \\ K &= -2H(b_k v^k - H\, v^k \Box v_k) \end{cases} \tag{A.9}$$

The D'Alembertian is the usual one, namely $\Box H = \eta^{ij} H_{;ij}$. It is easy to see that the Ricci scalar is given by

$$R = -2(b_k v^k + b\, v^k_{;k}) \ . \tag{A.10}$$

Without loss of generality we can take $v_t \equiv v_0 = 1$. This can always be achieved by rescaling H. Thus giving us

$$\begin{aligned} R_{tt} &= \Box H + K \\ R_{t\alpha} &= R_{tt} v_\alpha + C_\alpha \\ R_{\alpha\beta} &= R_{tt} v_\alpha v_\beta + C_\alpha v_\beta + C_\beta v_\alpha + A_{\alpha\beta} \end{aligned} \tag{A.11}$$

$$\text{where} \begin{cases} C_\alpha &= H \Box v_\alpha + 2\eta^{ij} H_i\, v_{\alpha;j} - b_\alpha \\ A_{\alpha\beta} &= 2\eta^{ij} H\, v_{\alpha;i} v_{\beta;j} - b(v_{\alpha;\beta} + v_{\beta;\alpha}) \end{cases} \tag{A.12}$$

In chapters two and three, we use covariant derivatives of the Killing vector fields to calculate the mass and angular momentum. In general if ξ^i is a Killing vector field then

$$\xi^{i|j} = \frac{1}{2} g^{ik} g^{jm} (\xi_{k,m} - \xi_{m,k}) \ . \tag{A.13}$$

Indeed if $\xi^i = \delta^i_A$ where A is just one of the coordinates then

$$\xi^{i|j} = \frac{1}{2} g^{ik} g^{jm} (g_{kA,m} - g_{mA,k}) \ . \tag{A.14}$$

For g_{ij} a Kerr-Schild metric we have

$$\begin{aligned}\xi^{i|j} &= \frac{1}{2}g^{ik}g^{jm}(\eta_{kA,m} - \eta_{mA,k}) + v_A(\eta^{jk}H_k v^i - \eta^{ik}H_k v^j) \\ &+\eta^{ik}\eta^{jm}H\, v_A(v_{k,m} - v_{m,k}) + H(v^i\eta^{jk} - v^j\eta^{ik})v_{A,k}\end{aligned} \tag{A.15}$$

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APPENDIX B

A CLASS OF TOROIDAL KERR-SCHILD METRICS

In chapters two and three, we are interested in the Ricci tensor of a particular class of Kerr-Schild metrics. The class of line element that we consider is

$$ds^2 = \eta_{ij}\, dx^i\, dx^j + 2H\,(v_i\, dx^i)^2 \tag{B.1}$$

where H is now taken to be a function of σ and ψ only, that is, $H = H(\psi, \sigma)$. The non-zero components of the η_{ij} metric are

$$\eta_{tt} = -1, \quad \eta_{\phi\phi} = \frac{a^2 sh^2 ch^2}{N^4}, \quad \eta_{\psi\psi} = \frac{a^2}{4N^4}, \quad \eta_{\sigma\sigma} = \frac{a^2}{4N^4} \tag{B.2}$$

and the components of the v_i are

$$v_t = 1, \quad v_\phi = \frac{a\, sh^2}{N^2}, \quad v_\psi = -\frac{a\, s}{2N^3}, \quad v_\sigma = -\frac{a}{2N^3}\frac{sh}{ch}c \tag{B.3}$$

with $N = \sqrt{s^2 + sh^2}$ where we have introduced the following notation:

$$\begin{aligned} c &:= \cos\frac{\psi}{2}, \quad & s &:= \sin\frac{\psi}{2} \\ ch &:= \cosh\frac{\sigma}{2}, \quad & sh &:= \sinh\frac{\sigma}{2}. \end{aligned} \tag{B.4}$$

The exterior and the interior metrics that we choose for both the Kerr and the Kerr-Newman sources in chapters two and three, respectively, are examples of this class of Kerr-Schild metrics, [see equations (2.4.28) and (3.2.8)].

The non zero covariant derivatives of the v_i are

$$\begin{aligned}
v_{\phi;\phi} &= \frac{ac}{N^3}sh^2 \quad , \quad v_{\phi;\sigma} = \frac{a}{2N^2}\frac{sh}{ch} \\
v_{\psi;\phi} &= \frac{a\,sc}{N^4}sh^2 \quad , \quad v_{\sigma;\phi} = \frac{a}{2N^4}\frac{sh}{ch}(c^2sh^2 - s^2ch^2) \\
v_{\psi;\psi} &= \frac{a\,c}{4N^5}sh^2 \quad , \quad v_{\psi;\sigma} = \frac{a\,s}{4N^3}\frac{sh}{ch}(1 - \frac{c^2}{N^2}) \\
v_{\sigma;\psi} &= -\frac{a\,s}{4N^5}sh\,ch \quad , \quad v_{\sigma;\sigma} = \frac{a\,c}{4N^3}\frac{sh^2}{ch^2} + \frac{a\,s^2c}{4N^5}
\end{aligned} \tag{B.5}$$

From the above we calculate $\Box v_i$ as

$$\Box v_t = 0, \quad \Box v_\phi = -\frac{2sh^2}{a}, \quad \Box v_\psi = \frac{s}{aN}, \quad \Box v_\sigma = \frac{c}{aN}\frac{sh}{ch} \tag{B.6}$$

Using (B.2), (B.3), and (B.6), we get

$$v^i \Box v_i = -\frac{2N^2}{a^2} \tag{B.7}$$

From (A.9) and (B.5) we obtain

$$b = \frac{2N}{a}\,c\,H - \frac{2N}{a}\,s\,H_\psi - \frac{2N}{a}\,c\,\frac{sh}{ch}\,H_\sigma \tag{B.8}$$

From (A.9) and (B.6) we get

$$\begin{aligned}
K = &-\frac{8N^2}{a^2}\,s^2\,H\,H_{\psi\psi} - \frac{16N^2}{a^2}\,sc\,\frac{sh}{ch}\,H\,H_{\psi\sigma} - \frac{8N^2}{a^2}\,c^2\,\frac{sh^2}{ch^2}\,H\,H_{\sigma\sigma} \\
&-\frac{8N^2}{a^2}\,s^2\,H^2 + \frac{4N^4}{a^2}\,\frac{sh}{ch^3}\,H\,H_\sigma
\end{aligned} \tag{B.9}$$

We obtain $\Box H$ as follows

$$\begin{aligned}
\Box H &= \eta^{ij}\,H_{;ij} = \frac{1}{\sqrt{-\eta}}\partial_i(\sqrt{-\eta}\,\eta^{ij}\,H_j) \\
&= \frac{4N^4}{a^2}(H_{\psi\psi} + H_{\sigma\sigma}) - \frac{4N^2}{a^2}\,sc\,H_\psi + \frac{2N^2}{a^2}\,\frac{1}{shch}(s^2ch^2 - c^2sh^2)H_\sigma
\end{aligned} \tag{B.10}$$

The functions C_α and $A_{\alpha\beta}$ as given by equation (A.12) are

$$\begin{aligned}
C_\phi &= -\frac{2sh^2}{a}H + \frac{4N^2}{a}\frac{sh}{ch}H_\sigma \\
C_\psi &= \frac{2N}{a}s\,H_{\psi\psi} + \frac{2N}{a}c\frac{sh}{ch}H_{\sigma\psi} + \frac{s}{aN}(s^2+N^2)H \\
&\quad + \frac{c}{aN}sh^2\,H_\psi + \frac{s}{aN}\frac{sh}{ch}(N^2-c^2)H_\sigma \\
C_\sigma &= \frac{2N}{a}c\frac{sh}{ch}H_{\sigma\sigma} + \frac{2N}{a}s\,H_{\psi\sigma} - \frac{c}{aN}\frac{sh^3}{ch}H \\
&\quad - \frac{s}{aN}sh\,ch\,H_\psi + \frac{\dot{c}}{aN}\frac{1}{ch^2}(N^2sh^2+ch^2s^2)H_\sigma
\end{aligned} \tag{B.11}$$

and

$$\begin{aligned}
A_{\phi\phi} &= \frac{2sh^2}{N^2}B \;,\; A_{\phi\psi} = \frac{s}{N^3}sh^2\,B \;,\; A_{\phi\sigma} = \frac{c}{N^3}\frac{sh^3}{ch}B \\
A_{\psi\psi} &= \frac{sh^2}{2N^4}B \;,\; A_{\psi\sigma} = -\frac{s\,c}{2N^4}\frac{sh}{ch}B \;,\; A_{\sigma\sigma} = \frac{1}{2N^4}(s^2+N^2\frac{sh^2}{ch^2})B
\end{aligned} \tag{B.12}$$

where

$$B = H - \frac{a\,c}{N}b \;. \tag{B.13}$$

We can extend C_α and $A_{\alpha\beta}$ so that they are defined for all the coordinates. Using equations (A.12) and (B.5) we get the trivial results $C_t = A_{tt} = A_{t\alpha} = 0$. The R^t_A components of the Ricci tensor are required to calculate the contribution of the interior region to the mass and angular momentum of the source.

$$\begin{aligned}
R^t_A &= g^{ti}(v_A v_i R_{tt} + C_A v_i + C_i v_A + A_{Ai}) \\
&= -C_A + v_A(-R_{tt} + g^{ti}C_i) + g^{ti}A_{Ai}
\end{aligned} \tag{B.14}$$

$$\text{But } g^{ti}A_{Ai} = (\eta^{ti} + 2H\,v^i)(2\eta^{jk}\,H\,v_{A;j}v_{i;k} - b(v_{A;i} + v_{i;A})) = 0 \tag{B.15}$$

$$\text{And } g^{ti}C_i = (\eta^{ti} + 2H\,v^i)(H\,\Box v_i + 2\eta^{jk}H_j\,v_{i;k} - b_i) \tag{B.16}$$

$$= 2H^2 v^i\,\Box v_i - 2H\,v^i b_i = K$$

So putting (A.11),(B.15) and (B.16) into equation (B.14) we have

$$R^i_A = -(C_A + v_A\,\Box H)\ . \tag{B.17}$$

APPENDIX C

TABLE OF INTEGRALS

We have found it quite useful to catelog the integrals that we have used most extensively with the aim of cleaning up one or two of the calculations that appear in chapters two and three of this thesis. For brevity's sake we will only quote the results. The first set of integrals consists of integrations with respect to the variable ψ that occur frequently in our calculations. Most of the integrals can be done by using a double change of variable; first, we let $x = \tan\frac{\psi}{2}$ and then we make the change of variable $x = \tan y \tanh\frac{\sigma}{2}$. We remind the reader that $N^2 = \cosh^2\frac{\sigma}{2} - \cos^2\frac{\psi}{2}$. The integrals are as follows:

$$\int_{-\pi}^{\pi} \frac{d\psi}{N^2} = \frac{2\pi}{\sinh\frac{\sigma}{2}\cosh\frac{\sigma}{2}} \tag{C.1}$$

$$\int_{-\pi}^{\pi} \frac{d\psi}{N^4} = \frac{\pi}{\cosh^3\frac{\sigma}{2}\sinh^3\frac{\sigma}{2}}\left(\cosh^2\frac{\sigma}{2} + \sinh^2\frac{\sigma}{2}\right) \tag{C.2}$$

$$\int_{-\pi}^{\pi} d\psi \frac{\sin^2\frac{\psi}{2}}{N^4} = \frac{\pi}{\sinh\frac{\sigma}{2}\cosh^3\frac{\sigma}{2}} \tag{C.3}$$

$$\int_{-\pi}^{\pi} d\psi \frac{\cos^2\frac{\psi}{2}}{N^4} = \frac{\pi}{\sinh^3\frac{\sigma}{2}\cosh\frac{\sigma}{2}} \tag{C.4}$$

$$\int_{-\pi}^{\pi} \frac{d\psi}{N^6} = \frac{\pi}{4\cosh^5\frac{\sigma}{2}\sinh^5\frac{\sigma}{2}}\left(3\cosh^4\frac{\sigma}{2} + 2\sinh^2\frac{\sigma}{2}\cosh^2\frac{\sigma}{2} + 3\sinh^4\frac{\sigma}{2}\right). \tag{C.5}$$

$$\int_{-\pi}^{\pi} d\psi \frac{\sin^2\frac{\psi}{2}}{N^6} = \frac{\pi}{\cosh^5\frac{\sigma}{2}\sinh^3\frac{\sigma}{2}}\left(\cosh^2\frac{\sigma}{2} + 3\sinh^2\frac{\sigma}{2}\right) \tag{C.6}$$

$$\int_{-\pi}^{\pi} d\psi \frac{\cos^2\frac{\psi}{2}}{N^6} = \frac{\pi}{\cosh^3\frac{\sigma}{2}\sinh^5\frac{\sigma}{2}}\left(3\cosh^2\frac{\sigma}{2} + \sinh^2\frac{\sigma}{2}\right). \tag{C.7}$$

Another set of integrals that occur quite regularly are

$$\int_{-\pi}^{\pi} d\psi \frac{\cos\frac{\psi}{2}}{N^n} \quad \text{and} \quad \int_{-\pi}^{\pi} d\psi \frac{(\cos\frac{\psi}{2})^{2m+1}}{N^n} \tag{C.8}$$

where n and m are positive integers. Using the same substitution described earlier in this appendix, the former of these integrals can be reduced to

$$\int_{-\pi}^{\pi} d\psi \frac{\cos\frac{\psi}{2}}{N^n} = \frac{4}{(\sinh\frac{\sigma}{2})^{n-1}} \int_{0}^{(\cosh\frac{\sigma}{2})^{-1}} dy\,(\cos y)^{n-2} \quad . \tag{C.9}$$

The integral on the right-hand side of (C.9) is a standard integral that can be performed quite easily. The latter integral can be evaluated using the iteration formula

$$\int_{-\pi}^{\pi} d\psi \frac{(\cos\frac{\psi}{2})^{2m+1}}{N^n} = \cosh^2\frac{\sigma}{2} \int_{-\pi}^{\pi} d\psi \frac{(\cos\frac{\psi}{2})^{2m-1}}{N^n} - \int_{-\pi}^{\pi} d\psi \frac{(\cos\frac{\psi}{2})^{2m-1}}{N^{n-2}} \quad . \tag{C.10}$$

Specific examples of these types of integrals that occur in the text of the thesis are

$$\int_{-\pi}^{\pi} d\psi \frac{\cos\frac{\psi}{2}}{N^3} = \frac{4}{\sinh^2\frac{\sigma}{2}\cosh\frac{\sigma}{2}} \tag{C.11}$$

$$\int_{-\pi}^{\pi} d\psi \frac{\cos\frac{\psi}{2}}{N^5} = \frac{4}{3}\frac{1}{\sinh^4\frac{\sigma}{2}\cosh^3\frac{\sigma}{2}}[3\cosh^2 - 1] \tag{C.12}$$

$$\int_{-\pi}^{\pi} d\psi \frac{\cos\frac{\psi}{2}}{N^7} = \frac{4}{\sinh^6\frac{\sigma}{2}\cosh\frac{\sigma}{2}}[1 - \frac{2}{3}\frac{1}{\cosh^2\frac{\sigma}{2}} + \frac{1}{5}\frac{1}{\cosh^4\frac{\sigma}{2}}] \quad . \tag{C.13}$$

The result that is employed most often when integrating over the σ variable is simple integration by parts formula,

$$\int_{\sigma_1}^{\sigma_2} d\sigma \frac{f}{N^{n+2}} \sinh\frac{\sigma}{2}\cosh\frac{\sigma}{2} = [-\frac{2}{n}\frac{f}{N^n}]_{\sigma_1}^{\sigma_2} + \frac{2}{n}\int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{N^n}\frac{\partial f}{\partial\sigma} \quad . \tag{C.14}$$

The final set of valuable integrals are the iteration formulae

$$\int \frac{x^2\,dx}{(x^2-1)^2} = \frac{-x}{x^2-1} - \int \frac{dx}{(x^2-1)^2} \tag{C.15}$$

$$\int \frac{dx}{x^2(x^2-1)^2} = \frac{1}{x(x^2-1)} + 3\int \frac{dx}{(x^2-1)^2} \tag{C.16}$$

$$\int \frac{dx}{x^4(x^2-1)^2} = \frac{1}{3x^3(x^2-1)} + \frac{5}{3}\int \frac{dx}{x^2(x^2-1)^2} \; . \tag{C.17}$$

APPENDIX D

PROJECTED COMPONENTS OF THE RICCI TENSOR

In chapter four, we demonstrated how to derive the projected components of the Ricci tensor. The full form of the projected components of the Ricci tensor are given below. The equations (4.3.73), (4.3.74), and (4.3.75) can be expanded by using the generalised Gauss-Codazzi equations (4.3.64) $\cdots$ (4.3.69) to give

$$\begin{aligned}\hat{R}_{ab} &= R_{ab} - K^A K_{Aab} + K_{Aa}{}^c K^A{}_{bc} - K^A{}_{ab\|A} \\ &\quad + K^A{}_b{}^c \mathbb{K}_{Aca} - \mathbb{P}_{aA}{}^A{}_{;b} - \mathbb{P}_{aA}{}^B P_{bB}{}^A \\ &\quad + 2\gamma^c_A \{K^A{}_{a[b;c]} - K^B{}_{a[c} P_{b]B}{}^A\} \end{aligned} \tag{D.1}$$

where $K^A = \gamma^{ab} K^A{}_{ab}$.

$$\begin{aligned}\hat{R}_a{}^A &= K^{Ab}{}_{a;b} - K^A{}_{,a} - K^B P_{aB}{}^A + K^B{}_{ba} P^b{}_B{}^A \\ &\quad + \gamma^{BD} \mathbb{\Gamma}^A_{BD,a} - P_a{}^{BA}{}_{\|B} + K^A{}_a{}^c \mathbb{P}_{cB}{}^B - \mathbb{P}^b{}_B{}^A K^B{}_{ab} \\ &\quad + \gamma^{Bb} \{-2\mathbb{K}_{Bc[a} K^A{}_{b]}{}^c + \mathbb{P}_{aA}{}^A{}_{;b} + \mathbb{P}_{bA}{}^A{}_{;a} + 2P_{[b}{}^{DA} \mathbb{P}_{a]BD}\} \\ &\quad + 2\gamma^b_B \gamma^{Bd} \{K^A{}_{d[a;b]} - K^D{}_{d[b} P_{a]D}{}^A\} \ . \end{aligned} \tag{D.2}$$

And finally we have

$$\begin{aligned}\hat{R}^{AB} &= \mathbb{R}^{AB} - \mathbb{P}_a{}^{BA} \mathbb{P}^a + \mathbb{P}_{aD}{}^A \mathbb{P}^{aBD} - K^{A\|B} \\ &\quad - \mathbb{K}^{Bab} K^A{}_{ab} - \mathbb{P}^{aBA}{}_{;a} - 2P^{a(BD)} \mathbb{P}_{aD}{}^A - P^{aDA} \mathbb{P}_a{}^B{}_D \\ &\quad - \gamma^{Ba} \{\gamma^{DF} \mathbb{\Gamma}^A_{DE,a} - P_a{}^{DA}{}_{\|D} + K^{Ab}{}_{a;b} - K^A{}_{,a} \\ &\quad + K^A{}_{ab} \mathbb{P}^b - K^D P_{aD}{}^A - K_{Dab} \mathbb{P}^{bDA} + K_{Dab} P^{bDA}\} \end{aligned}$$

$$+\gamma^{a}_{D}\{\gamma^{BE}\gamma^{DF}\,\mathbb{\Gamma}^{A}_{EF,a} - P_{a}{}^{DA\|B} + K^{A}{}_{ac}\,\mathbb{P}^{cBD} - \mathbb{P}^{bBA}\,K^{D}{}_{ab}\}$$
$$+\gamma^{Da}\gamma^{Bb}\{-2\,\mathbb{K}_{Dc[a}\,K^{A}{}_{b]}{}^{c} + \mathbb{P}_{aD}{}^{A}{}_{;b} + \mathbb{P}_{bD}{}^{A}{}_{;a} - 2\,P_{[a}{}^{EA}\,\mathbb{P}_{b]DE}\}$$
$$-2\,\gamma^{c}_{D}\,\gamma^{Da}\,\gamma^{Bb}\{K^{A}{}_{a[b;c]} - K^{E}{}_{a[c}\,P_{b]E}{}^{A}\} \qquad \text{(D.3)}$$

where $\mathbb{P}^{a} = \mathbb{P}^{a}{}_{A}{}^{A}$ and $\mathbb{R}^{AB} = \gamma^{AD}\,\gamma^{BE}\,\mathbb{R}^{F}{}_{DFE}$.